4 Calculus

Summary

Calculus is concerned with two basic operations, differentiation and integration, and is a tool used by engineers to determine such quantities as rates of change and areas; in fact, calculus is the mathematical 'backbone' for dealing with problems where variables change with time or some other reference variable and a basic understanding of calculus is essential for further study and the development of confidence in solving practical engineering problems. This will become evident in the next chapter where physical systems will be modelled and the use of 'rates of change' equations (called differential equations) will allow the physical system to be represented, an analysis made and a solution formed under defined conditions. This chapter is an introduction to the techniques of calculus and a consideration of some of their engineering applications. The topic continues in the next chapter with a discussion of the use of differential equations to represent physical systems and their solution for various inputs.

Objectives

By the end of this chapter, the reader should be able to:

- understand the concept of a limit and its significance in rate of change relationships;
- use calculus notation for describing a rate of change (differentiation) and understand the significance of the operation;
- solve engineering problems involving rates of change;
- understand what is involved in the calculus operation of integration;
- solve engineering problems involving integration.

4.1 Differentiation

Suppose we have an equation describing how the distance covered in a straight line by a moving object varies with time. We could plot a graph of displacement against time and determine the velocity at some instant as the gradient of the tangent to the curve at that instant. By taking a number of such gradient measurements we could then determine how the velocity varied with time. However, *differentiation* is a mathematical technique which can be used to determine the rate at which functions change and hence the gradients, this thus enabling the velocity to be obtained from the equation without drawing the graph and tangents. We could

Key point

Differentiation is a mathematical technique which can be used to determine the rate at which functions change and hence the gradients of the tangents to graphs.

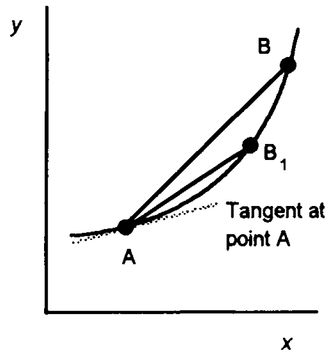


Figure 4.1 *Gradient at A*

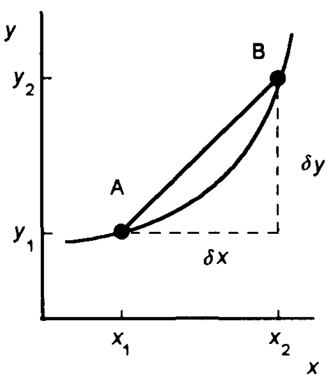


Figure 4.2 *Gradient*

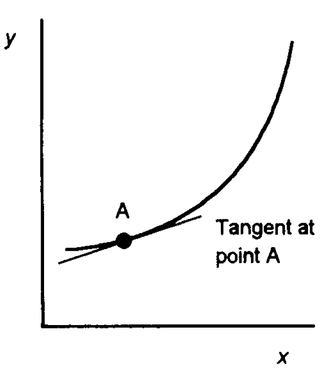


Figure 4.3 *Gradient at A*

also, for example, describe how the deflection of an initially horizontal beam in the y-direction alters with distance along the beam in the x-direction, or how the volume of a gas changes with temperature, or how electric charge at a point in a circuit changes with time (called current), etc. The list of uses is endless!

Limits

Consider the problem of determining the gradient of a tangent at a point on a graph. It might, for example, be a distance–time graph for a moving object to determine velocity as the rate at which distance is covered or a current–time graph for the current in an electrical circuit in order to determine the rate of change of current with time. Suppose we want to determine the gradient at point A on the curve shown in Figure 4.1. We can select another point B on the curve and join them together and then find the gradient of the line AB. The value of the gradient determined in this way will depend on where we locate the point B. If we let B slide along the curve towards A then the closer B is to A the more the line approximates to the tangent at point A. Thus the line AB_1 is a better approximation to the tangent than the line AB. The method we can use to determine the gradient of the tangent at point A is:

1. Take another point B on the same curve and determine the gradient of the line joining A and B.
2. Then move B closer and closer to A. In the limit, as the distance between A and B becomes infinitesimally small, i.e. as AB tends to zero (written as AB $\to$ 0), the gradient of the line becomes the gradient of the tangent at A.

Consider the gradient of the line AB in Figure 4.2:

$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}$$

The gradient is the difference in the value of y between points A and B divided by the difference in the value of x between the points. We can write this difference in the value of x as δx and this difference in the value of y as δy. The δ symbol in front of a quantity means 'a small bit of it' or 'an interval of'. Thus the equation can be written as

$$\text{gradient} = \frac{\delta y}{\delta x}$$

An alternative symbol which is often used is Δx, with the Δ symbol being used to indicate that we are referring to a small bit of the quantity x. These forms of notation do *not* mean that we have δ or Δ multiplying x. The δx or Δx should be considered as a single symbol representing a single quantity.

As we move B closer to A then the interval δx is made smaller. The gradient of the line AB then becomes closer to the tangent to the curve at point A (Figure 4.3). Eventually when the difference

in x between A and B, i.e. δx, tends to zero then we have the gradient of the tangent at point A. We can write this as:

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} \qquad [1]$$

This reads as: the limiting value of $\delta y/\delta x$ as δx tends to a zero value equals dy/dx. A *limit* is a value to which we get closer and closer as we carry out some operation. Thus dy/dx is the value of the gradient of the tangent to the curve at A. Since the tangent is the instantaneous rate of change of y with x at that point then dy/dx is the instantaneous rate of change of y with respect to x. dy/dx is called the *derivative* of y with respect to x. The process of determining the derivative for a function is called *differentiation*. The notation dy/dx should not be considered as d multiplied by y divided by d multiplied by x, but as a single symbol representing the gradient of the tangent and so the rate of change of y with x; if you like, it is a shorthand way of writing 'the rate of change of y with respect to x'.

Since we can interpret the derivative as representing the slope of the tangent to a graph of a function at a particular point, this means with a continuous function, i.e. a function which has values of y which smoothly and continuously change as x changes for all values of x, that we have derivatives for all values of x. However, with a discontinuous graph there will be some values of x for which we can have no derivative. For example, with the graph shown in Figure 4.4, there is no derivative for $x = 1$.

Key point

For any graph of $y = f(x)$, if A is the point (x, y), i.e. $(x, f(x))$, and B the point $(x + dx, y + \delta y)$, i.e. $(x + \delta x, f(x + \delta x))$, then the gradient of the tangent at point P at the limiting value as the δx tends to zero:

$$\text{slope of tangent} = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

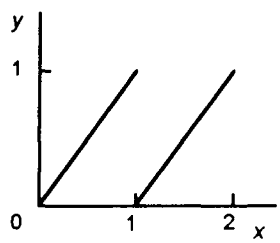


Figure 4.4 *Discontinuous function*

Example

Determine the slope of the tangent to the curve $y = f(x) = x^2$ when we have $x = 1$ and $y = 1$.

$$\text{Slope of tangent} = \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{x^2 + 2x(\delta x) + (\delta x)^2 - x^2}{\delta x}$$

$$= \lim_{\delta x \to 0} (2x + \delta x)$$

As δx tends to zero, the slope of the tangent tends to the value $2x$. With $x = 1$ then the slope of the tangent is 2.

Maths in action

There are situations in engineering where, given an initial condition of some variable, we need to determine its rate of change with respect to some parameter.

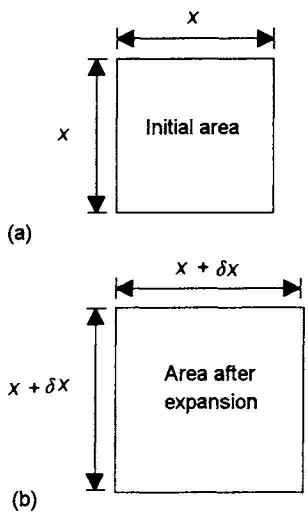


Figure 4.5 *Expansion of a plate*

As an illustration, consider a square flat metal plate of side length x (Figure 4.5(a)). The initial condition we are concerned with is the area A of the plate:

initial area $A = x$ multiplied by $x = x^2$

Now suppose the plate is heated and the dimension x changes by an amount δx (which is small compared with the original dimension x but which may have an effect on such things as tolerances in assembly). The plate, which expands equally in all directions, now has sides of length $(x + \delta x)$ (Figure 4.5(b)). The new area is thus:

new area $= (x + \delta x)(x + \delta x) = x^2 + 2x(\delta x) + (\delta x)^2$

If we denote the changes in area as δA, then:

$\delta A =$ new area – initial area $= x^2 + 2x(\delta x) + (\delta x)^2 - x^2$

$= 2x(\delta x) + (\delta x)^2$

Since dx is very small, then $(\delta x)^2 \to 0$ and in this limiting condition we can write:

$$\mathrm{d}A = 2x\,\mathrm{d}x$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x$$

We now have an expression which describes how the rate of change of the area with side length depends on the side length. We have determined the derivative.

4.1.1 Derivatives of common functions

The above examples illustrate how, given some initial condition, we can determine the derivative of a function. Rather than always work from first principles in this way, it is useful to work out some general rules we can use. The following illustrates how some commonly used functions can be differentiated.

Derivative of a constant

Key point

The derivative of a constant is zero.

A graph of a constant, e.g. $y = 2$, has a gradient of 0. Thus its derivative is zero. Thus for $y = c$, where c is a constant:

$$\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0 \qquad [2]$$

Key point

$\frac{d}{dx}(x^n) = nx^{n-1}$

Derivative of x^n

If we differentiate from first principles $y = x$ we obtain $dy/dx = 1$. If we differentiate $y = x^2$, as in the above example and maths in action, we obtain $dy/dx = 2x$. If we differentiate $y = x^3$ we obtain $dy/dx = 3x^2$. The pattern in these differentiations is that if we have $y = x^n$, then:

$$\frac{d}{dx}(x^n) = nx^{n-1} \qquad [3]$$

This relationship applies for positive, negative and fractional values of n.

Example

Determine the derivative of the functions (a) $y = x^{3/2}$, (b) $y = x^{-4}$.

(a) $\frac{dy}{dx} = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}}$

(b) $\frac{dy}{dx} = -4x^{-4-1} = -4x^{-5}$

Derivatives of trigonometric functions

Consider the determination of how the gradients of the graph of $y = \sin x$ (Figure 4.6(a)) vary with x. Examination of the graphs shows that:

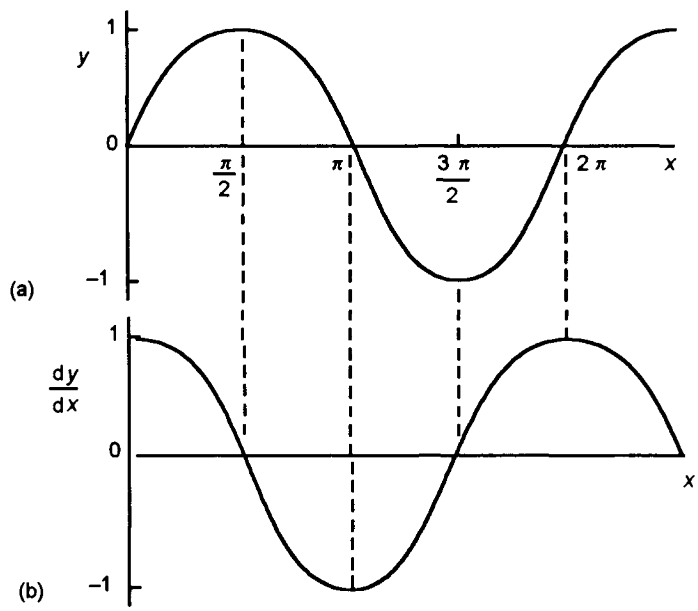


Figure 4.6 *Gradients of $y = \sin x$*

1 As x increases from 0 to $\pi/2$ then the gradient, which is positive, gradually decreases to become zero at $x = \pi/2$.

2 As x increases from $\pi/2$ to π then the gradient, which is now negative, becomes steeper and steeper to reach a maximum value at $x = \pi$.

3 As x increases from π to $3\pi/2$ then the gradient, which is negative, decreases to become zero at $x = 3\pi/2$.

4 As x increases from $3\pi/2$ to 2π the gradient, which is now positive again, increases to become a maximum at $x = 2\pi$.

Figure 4.6(b) shows the result that is obtained by plotting the gradients against x; it is a cosine curve. Thus, the derivative of $y = \sin x$ is:

$$\frac{d}{dx}(\sin x) = \cos x \qquad [4]$$

We can prove that the above is the case as follows. For the function $f(x) = \sin x$, we have $f(x + \delta x) - f(x) = \sin(x + \delta x) - \sin x$. Using equation [28] from Chapter 1 for the sum of two angles, $\sin(x + \delta x) = \sin x \cos \delta x + \cos x \sin \delta x$. As δx tends to 0 then $\cos \delta x$ tends to 1 and $\sin \delta x$ to δx. Thus, the derivative can be written as:

$$\frac{d}{dx}(\sin x) = \lim_{\delta x \to 0} \frac{\sin x + \delta x \cos x}{\delta x} = \cos x$$

If we had considered the function $\sin ax$ then we would have obtained:

$$\frac{d}{dx}(\sin ax) = a \cos ax \qquad [5]$$

In a similar manner we can consider $y = \cos x$ (Figure 4.7(a)) and the gradients at various points along the graph.

1 Between $x = 0$ and $x = \pi/2$ the gradient, which is negative, becomes steeper and steeper and reaches a maximum value at $x = \pi/2$.

2 Between $x = \pi/2$ and $x = \pi$ the gradient, which is negative decreases until it becomes zero at $x = \pi$.

3 Between $x = \pi$ and $x = 3\pi/2$ the gradient, which is positive, increases until it becomes a maximum at $x = 3\pi/2$.

4 Between $x = 3\pi/2$ and $x = 2\pi$ the gradient, which is positive, decreases to become zero at $x = 2\pi$.

Figure 4.7(b) shows how the gradient varies with x. The result is an inverted sine graph. Thus, for $y = \cos x$:

$$\frac{d}{dx}(\cos x) = -\sin x \qquad [6]$$

Key points

$$\frac{d}{dx}(\sin ax) = a \cos ax$$

$$\frac{d}{dx}(\cos ax) = -a \sin ax$$

The derivatives of tan ax, cosec ax, sec ax and cot ax can be derived using the quotient rule (see later in this chapter) as:

$$\frac{d}{dx}(\tan ax) = a \sec^2 ax$$

$$\frac{d}{dx}(\operatorname{cosec} ax) = -a \operatorname{cosec} ax \cot ax$$

$$\frac{d}{dx}(\sec ax) = a \sec ax \tan ax$$

$$\frac{d}{dx}(\cot ax) = -a \operatorname{cosec}^2 ax$$

The derivatives of sin $(ax + b)$, cos $(ax + b)$, etc. can be derived by using the chain rule (see later in this chapter) as:

$$\frac{d}{dx}[\sin(ax+b)] = a \cos(ax+b)$$

$$\frac{d}{dx}[\cos(ax+b)] = -a \sin(ax+b)$$

$$\frac{d}{dx}[\tan(ax+b)] = a \sec^2(ax+b)$$

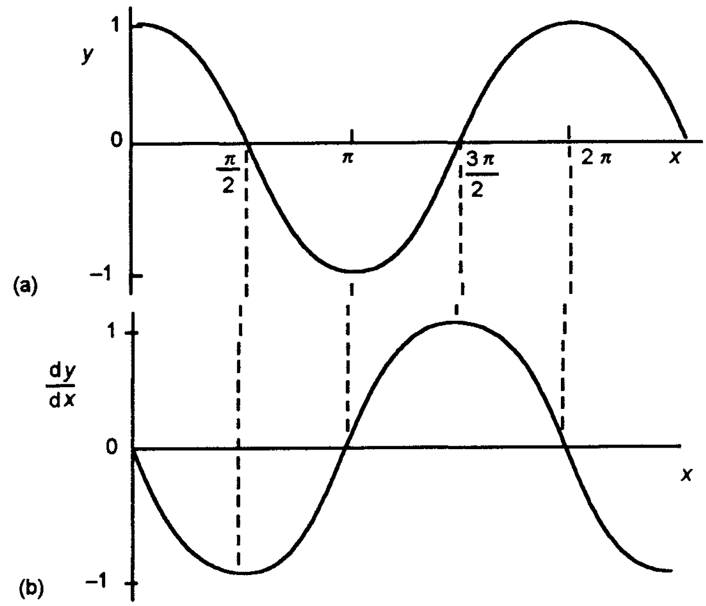


Figure 4.7 *Gradients of y = cos x*

If we had considered $y = \cos ax$, where a is a constant, then we would have obtained for the derivative:

$$\frac{d}{dx}(\cos ax) = -a \sin ax \qquad [7]$$

We can prove that the above is the case in a similar way to that used for the sine.

Example

Determine the derivatives of (a) sin 2*x*, (b) cos 3*x*.

(a) $\frac{d}{dx}(\sin 2x) = 2\cos 2x$, (b) $\frac{d}{dx}(\cos 3x) = -3\sin 3x$

Maths in action

Consider a sinusoidal current $i = I_m \sin \omega t$ passing through a *pure inductance*. A pure inductance is one which has only inductance and no resistance or capacitance. With an inductance a changing current produces a back e.m.f. of $-L \times$ the rate of change of current, i.e. $L\, di/dt$, where L is the inductance. The applied e.m.f. must overcome this back e.m.f. for a current to flow. Thus the voltage across the inductance is $L\, di/dt$. Hence:

$$v = L\frac{di}{dt} = L\frac{d}{dt}(I_m \sin \omega t) = \omega L I_m \cos \omega t$$

Since cos ωt = sin (ωt + 90°), the current and the voltage are out of phase with the voltage leading the current by 90°.

Consider a circuit having just *pure capacitance* with a sinusoidal voltage $v = V_m \sin \omega t$ being applied across it. A pure capacitance is one which has only capacitance and no resistance or inductance. The charge q on the plates of a capacitor is related to the voltage v by $q = Cv$. Thus, since current is the rate of movement of charge dq/dt:

i = rate of change of q = rate of change of (Cv)

= C × (rate of change of v)

i.e. $i = C\,dv/dt$. Since current is the rate of change of charge q:

$$i = \frac{dq}{dt} = \frac{d}{dt}(Cv) = C\frac{d}{dt}(V_m \sin \omega t) = \omega C V_m \cos \omega t$$

Since cos ωt = sin (ωt + 90°), the current and the voltage are out of phase, the current leading the voltage by 90°.

Derivatives of exponential functions

Consider the exponential equation $y = e^x$ and a small increase in x of δx. The corresponding increase in the value of y is δy where:

$$y + \delta y = e^{x+\delta x} = e^x e^{\delta x}$$

Thus:

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\text{change in } y \text{ when } x \text{ changes by } \delta x}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{e^x e^{\delta x} - e^x}{\delta x} = \lim_{\delta x \to 0} \frac{e^x(e^{\delta x} - 1)}{\delta x}$$

If we let $\delta x = 0.01$ then $(e^{0.01} - 1)/0.01 = 1.005$. If we take yet smaller values of dx then in the limit this has the value 1. Thus:

$$\frac{d}{dx}e^x = e^x \qquad [8]$$

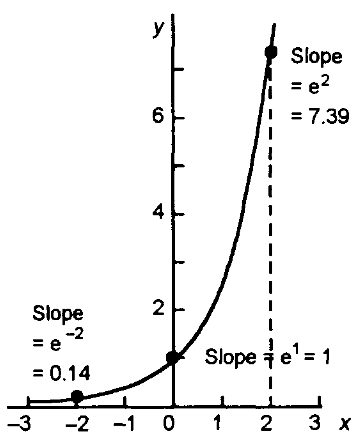


Figure 4.8 $y = e^x$

The derivative of e^x is e^x. Thus the gradient of the graph of $y = e^x$ at a point is equal to the value of y at that point (Figure 4.8). For example, at the point $x = 0$ on the graph the gradient is $y = e^0 = 1$. At $x = 2$ the gradient is $y = e^2 = 7.39$. At $x = -2$ the gradient is $y = e^{-2} = 0.14$.

If we had $y = e^{ax}$ then:

Key point

$\frac{d}{dx}(e^{ax}) = a e^{ax}$

$$\frac{d}{dx}(e^{ax}) = a e^{ax} \qquad [9]$$

Example

Determine the derivative of $y = e^{2x}$.

$$\frac{d}{dx}e^{2x} = 2\,e^{2x}$$

Example

The variation of current i with time t in an electrical circuit is given by the equation $i = \sin 314t$. Derive an equation for the rate of change of current with time.

$$\frac{di}{dt} = \frac{d}{dt}(\sin 314t) = 314 \cos 314t$$

Maths in action

The variation with time t of the displacement y of a system oscillating with simple harmonic motion is described by the equation:

$$y = A \sin \omega t$$

where A is the amplitude and ω the angular frequency. The linear velocity v is the rate of change if displacement with time, i.e. dy/dt, and so:

$$v = \frac{dy}{dt} = -A\omega \sin \omega t$$

The acceleration a is the rate of change of velocity, i.e. dv/dt, and so:

$$a = \frac{dv}{dt} = -A\omega^2 \cos \omega t = -A\omega^2 y$$

The acceleration is thus proportional to the displacement and the minus sign indicates that it is always in the opposite direction to that in which y increases, i.e. it is always directed towards the central rest position. This is the definition used for harmonic motion or cyclic motion which is referred to as *simple harmonic motion* or, for short, SHM.

4.1.2 Rules of differentiation

In this section the basic rules are developed for the differentiation of constant multiples, sums, products and quotients of functions and the chain rule for functions of functions.

Multiplication by a constant

Consider a multiple of some function, e.g. $cf(x)$ where c is a constant.

Key point

The derivative of some function multiplied by a constant is the same as the constant multiplying the derivative of the function.

$$\frac{\mathrm{d}}{\mathrm{d}x}cf(x) = \lim_{\delta x \to 0} \frac{cf(x+\delta x) - cf(x)}{\delta x}$$

$$= c \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = c\frac{\mathrm{d}}{\mathrm{d}x}f(x) \qquad [10]$$

The derivative of some function multiplied by a constant is the same as the constant multiplying the derivative of the function.

Example

Determine the derivatives of (a) $4x^2$, (b) 2 sin $3x$, (c) $y = \frac{1}{3\sqrt{x}}$.

(a) $\frac{\mathrm{d}}{\mathrm{d}x}4x^2 = 4 \times 2x = 8x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(2\sin 3x) = 2 \times 3\sin 3x = 6\sin 3x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3\sqrt{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{3}x^{-1/2}\right) = \frac{1}{3} \times \left(-\frac{1}{2}\right)x^{-3/2} = -\frac{1}{6}x^{-3/2}$

Sums of functions

Consider a function which can be considered to be a sum of a number of other functions, e.g. $y = f(x) + g(x)$:

Key point

The derivative of the sum of two differentiable functions is the sum of their derivatives.

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) + g(x)] = \lim_{\delta x \to 0} \frac{[f(x+\delta x) + g(x+\delta x)] - [f(x) + g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] + \lim_{\delta x \to 0}\left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mathrm{d}}{\mathrm{d}x}g(x) \qquad [11]$$

The derivative of the sum of two differentiable functions is the sum of their derivatives.

As an illustration, consider the differentiation of the hyperbolic function $y = \sinh x$. This function (see Section 1.8) can be written as $\frac{1}{2}(e^x - e^{-x})$. Thus:

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\tfrac{1}{2}(e^x - e^{-x}) = \tfrac{1}{2}(e^x + e^x) = \cosh x \qquad [12]$$

In a similar way we can differentiate sinh ax and cosh ax, obtaining:

$$\frac{d}{dx}(\sinh ax) = a\cosh ax \qquad [13]$$

$$\frac{d}{dx}(\cosh ax) = a\sinh ax \qquad [14]$$

Key points

$$\frac{d}{dx}(\sinh ax) = a\cosh ax$$

$$\frac{d}{dx}(\cosh ax) = a\sinh ax$$

The hyperbolic function tanh ax can be differentiated using the quotient rule (see later in this Section).

$$\frac{d}{dx}(\tanh ax) = a\frac{1}{\cosh^2 ax}$$

Example

Determine the derivatives of:

(a) $y = 2x^3 + x^2$, (b) $y = \sin x + \cos 2x$, (c) $y = e^{4x} + x$

(a) $\frac{dy}{dx} = 6x^2 + 2x$ (b) $\frac{dy}{dx} = \cos x - 2\sin 2x$

(c) $\frac{dy}{dx} = 4e^{4x} + 1$

The product rule

Consider a function $y = f(x)g(x)$ which is the product of two other differentiable functions, e.g. $y = x \sin x$:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{\delta x \to 0}\frac{f(x+\delta x)g(x+\delta x) - f(x)g(x)}{\delta x}$$

We can simplify this by adding and subtracting the same quantity to the numerator, namely $f(x + \delta x)g(x)$, to give:

$$\frac{d}{dx}[f(x)g(x)$$

$$= \lim_{\delta x \to 0}\frac{f(x+\delta x)g(x+\delta x) + f(x+\delta x)g(x) - f(x+\delta x)g(x) - f(x)g(x)}{\delta x}$$

$$= \lim_{\delta x \to 0}\left[f(x+\delta x)\frac{g(x+\delta x) - g(x)}{\delta x} + g(x)\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

$$= f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \qquad [15]$$

This is often written in terms of u and v, where $u = f(x)$ and $v = g(x)$:

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} \qquad [16]$$

Key point

The derivative of the product of two differentiable functions is the sum of the first function multiplied by the derivative of the second function and the second function multiplied by the derivative of the first function.

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example

Determine the derivatives of the following functions:

(a) $y = x \sin x$, (b) $y = x^2 e^{3x}$, (c) $y = (2 + x)^2$, (d) $y = x e^x \sin x$

(a) $\dfrac{dy}{dx} = x\dfrac{d}{dx}(\sin x) + \sin x\dfrac{d}{dx}x = x\cos x + \sin x$

(b) $\dfrac{dy}{dx} = x^2\dfrac{d}{dx}e^{3x} + e^{3x}\dfrac{d}{dx}x^2 = 3x^2\,e^{3x} + 2x\,e^{3x}$

(c) This can be written as $(2 + x)(2 + x)$ and so:

$$\frac{dy}{dx} = (2+x)\frac{d}{dx}(2+x) + (2+x)\frac{d}{dx}(2+x) = 2(2+x)$$

(d) This product has three terms and so we have to carry out the differentiation in two stages. Thus, if we first consider $x\,e^x$ as one term and the $\sin x$ as the other term:

$$\frac{dy}{dx} = x\,e^x\frac{d}{dx}\sin x + \sin x\frac{d}{dx}(x\,e^x)$$

$$= x\,e^x\cos x + \sin x\frac{d}{dx}(x\,e^x)$$

We can then use the product rule to evaluate the derivative of $x\,e^x$.

$$\frac{d}{dx}(x\,e^x) = x\frac{d}{dx}e^x + e^x\frac{d}{dx}x = x\,e^x + e^x$$

Hence:

$$\frac{dy}{dx} = x\,e^x\cos x + x\,e^x + e^x$$

The quotient rule

Consider obtaining the derivative of a function which is the quotient of two other functions, e.g. $f(x)/g(x)$:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \lim_{\delta x\to 0}\frac{\dfrac{f(x+\delta x)}{g(x+\delta x)} - \dfrac{f(x)}{g(x)}}{\delta x}$$

$$= \lim_{\delta x\to 0}\frac{g(x)f(x+\delta x) - f(x)g(x+\delta x)}{\delta x g(x)g(x+\delta x)}$$

Adding and subtracting $f(x)g(x)$ to the numerator enables the above equation to be simplified:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \lim_{\delta x \to 0} \frac{g(x)f(x+\delta x) + f(x)g(x) - f(x)g(x) - f(x)g(x+\delta x)}{\delta x g(x)g(x+\delta x)}$$

$$= \frac{\lim_{\delta x \to 0} \frac{g(x)[f(x+\delta x) - f(x)]}{\delta x} - \lim_{\delta x \to 0} \frac{f(x)[g(x+\delta x) - g(x)]}{\delta x}}{\lim_{\delta x \to 0} [g(x)g(x+\delta x)]}$$

$$= \frac{g(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) - f(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x)}{[g(x)]^2} \quad [17]$$

Key point

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

This is often written in terms of u and v, where $u = f(x)$ and $v = g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \quad [18]$$

Note that if we have just the reciprocal of some function, i.e. $1/g(x)$, then we have $f(x) = 1$ and so equation [18] gives:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{g(x)}\right] = -\frac{1}{[g(x)]^2}\frac{\mathrm{d}}{\mathrm{d}x}g(x) \quad [19]$$

Equation [18] can be used to determine the derivative of tan x, since $\tan x = \sin x/\cos x$. Thus $f(x) = \sin x$ and $g(x) = \cos x$. Hence:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \frac{\cos x\frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x\frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \quad [20]$$

Likewise, equation [18] can be used to determine the derivative of tanh x.

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sinh x}{\cosh x}\right)$$

$$= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} \quad [21]$$

Example

Determine the derivative of $y = (2x^2 + 5x)/(x + 3)$.

Using equation [18] with $f(x) = 2x^2 + 5x$ and $g(x) = x + 3$:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x+3)(4x+5) - (2x^2+5x)(1)}{(x+3)^2} = \frac{2x^2+12x+1}{(x+3)^2}$$

Example

Determine the derivative of $y = x\,e^x/\cos x$.

This example requires the use of both the quotient and product rules for differentiation. Using equation [19] with $f(x) = x\,e^x$ and $g(x) = \cos x$:

$$\frac{dy}{dx} = \frac{\cos x\frac{d}{dx}(x\,e^x) - x\,e^x(-\sin x)}{\cos^2 x}$$

Now using equation [16] to obtain the derivative for the product $x\,e^x$:

$$\frac{dy}{dx} = \frac{\cos x(x\,e^x + e^x) - x\,e^x(-\sin x)}{\cos^2 x}$$

$$= \frac{e^x(x\cos x + \cos x + x\sin x)}{\cos^2 x}$$

The chain rule

Suppose we have $y = \cos x^4$ and, in order to differentiate it, write it in the form $y = \cos u$ and $u = x^4$. We can then obtain dy/du and du/dx, but how from them do we obtain dy/dx?

Consider the function $y = f(u)$ where $u = g(x)$ and the obtaining of the derivative of $y = f(g(x))$. For $u = g(x)$ a small increase of δx in the value of x causes a corresponding small increase of δu in the value of u. But $y = f(u)$ and so the small increase δu causes a correspondingly small increase of δy in the value of y. We can write, since the δu terms cancel:

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$$

Thus:

$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\right)\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

and so:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad [22]$$

Key point

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

This is known as the *function of a function rule* or the *chain rule*.

The chain rule can be used to determine the derivative a function such as $y = \sin x^n$, for n being positive or negative or fractional. Let $u = x^n$ and so consequently $y = \sin u$. Then $du/dx = nx^{n-1}$ and $dy/du = \cos u$. Hence, using the chain rule (equation [22]) we have $dy/dx = \cos u \times nx^{n-1} = nx^{n-1}\cos x^n$.

Another application of the chain rule is to determine the derivatives of functions of the form $y = (ax + b)^n$, $y = e^{ax+b}$, $y = \sin(ax + b)$, etc. With such functions we let $u = ax + b$ and so then we have, for the three examples, $y = u^n$, $y = e^u$, $y = \sin u$. Then we have $du/dx = a$ and $dy/du = nx^{n-1}$, $dy/du = e^u$, $dy/du = \cos u$. Using the chain rule we then obtain $dy/dx = anu^{n-1}$, $dy/dx = a\,e^{ax+b}$ and $dy/dx = a \cos u$. Thus, for the three examples, we have:

$$\frac{dy}{dx} = an(ax+b)^{n-1}, \quad \frac{dy}{dx} = a\,e^{ax+b}, \quad \frac{dy}{dx} = a\cos(ax+b), \text{ etc.}$$

Example

Determine the derivative of $y = (2x - 5)^4$.

Let $u = 2x - 5$ and so $y = u^4$. Then $du/dx = 2$ and $dy/du = 4u^3$ and so, using equation [22]:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4u^3 \times 2 = 8u^3 = 8(2x-5)^3$$

Example

Determine the derivative of $y = \sin x^3$.

Let $u = x^3$ and so $y = \sin u$. Then $dy/du = 3x^2$ and $dy/du = \cos u$ and so, using equation [22]:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3x^2 \times \cos u = 3x^2 \cos x^3$$

Example

Determine the derivative of $y = \sqrt{\dfrac{x^2}{x^2+1}}$.

Let $u = x^2/(x^2 + 1)$ and so $y = u^{1/2}$. Using the quotient rule:

$$\frac{du}{dx} = \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$$

Using the chain rule:

$$\frac{dy}{dx} = \tfrac{1}{2}u^{-1/2} \times \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$$

$$= \tfrac{1}{2}\left(\frac{x^2}{x^2+1}\right)^{-1/2} \frac{(x^2+1)2x - x^2(2x)}{(x^2+1)^2}$$

$$= \frac{x^{1/2}}{(x^2+1)^{3/2}} = \sqrt{\frac{x}{(x^2+1)^3}}$$

4.1.3 Higher-order derivatives

Consider a moving object for which we have a relationship between the displacement s of the object and time t of the form:

$$s = ut + \tfrac{1}{2}at^2$$

where u and a are constants. We can plot this equation to give a distance–time graph. If we differentiate this equation we obtain:

$$\frac{ds}{dt} = u + at$$

ds/dt is the gradient of the distance–time graph. It also happens to be the velocity. The gradient varies with time. We could thus plot a velocity–time graph, i.e. a ds/dt graph against t. Then differentiating for a second time, to obtain the gradients to this graph, we obtain the acceleration a.

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = a$$

The derivative of a derivative is called the *second derivative* and can be written as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } \frac{d^2y}{dx^2}$$

The first derivative gives information about how the gradients of the tangents change. The second derivative gives information about the rate of change of the gradient of the tangents.

If the second derivative is then differentiated we obtain the third derivative.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \text{ or } \frac{d^3y}{dx^3}$$

This, in turn, may be differentiated to give a fourth derivative, and so on.

$$\frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) \text{ or } \frac{d^4y}{dx^4}$$

Example

Determine the second derivative of $y = x^3$.

The first derivative is:

$$\frac{dy}{dx} = 3x^2$$

The second derivative is given by differentiating this equation again:

$$\frac{d^2y}{dx^2} = 6x$$

Example

Determine the second derivative of $y = x^4 + 3x^2$.

The first derivative is

$$\frac{dy}{dx} = 4x^3 + 6x$$

The second derivative is

$$\frac{d^2y}{dx^2} = 12x^2 + 6$$

Maths in action

This illustrates how differential calculus may be used in the analysis of a beam which is deflected in one plane as a result of loading. Consider a beam which is bent into a circular arc and the radius R of the arc. For a segment of circular arc (Figure 4.9), the angle $\delta\theta$ subtended at the centre is related to the arc length δs by $\delta s = R\delta\theta$. Because the deflections obtained with beams are small δx is a reasonable approximation to δs and so we can write $\delta x = R\delta\theta$ and $1/R = \delta\theta/\delta x$. The slope of the straight line joining the two end points of the arc is $\delta y/\delta x$ and thus $\tan \delta\theta = \delta y/\delta x$. Since the angle will be small we can make the approximation that $\delta\theta = \delta y/\delta x$. Hence:

$$\frac{1}{R} = \frac{\delta\theta}{\delta x} = \frac{\delta}{\delta x}\left(\frac{\delta y}{\delta x}\right)$$

In the limit we can thus write:

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

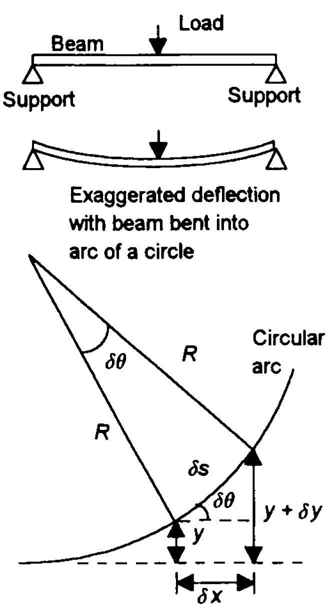


Figure 4.9 *The deflection curve of radius R*

When a beam is bent as a result of the application of a bending moment M it curves with a radius R given by the general bending equation as:

$$\frac{1}{R} = \frac{M}{EI}$$

where E is the modulus of elasticity and I the second moment of area (a property of the shape of the beam) and so we can write:

$$M = EI\frac{d^2y}{dx^2}$$

This differential equation provides the means by which the deflections of beams can be determined.

In Section 4.1.3 the determination of maximum and minimum points is discussed. We can use the criteria for a maximum in order to determine the conditions necessary for the deflection of the beam to be a maximum. In Section 4.2 we then use the above differential equation, with the condition for maximum deflection, to determine the maximum deflection of a beam.

4.1.4 Maxima and minima

There are many situations in engineering where we need to establish maximum or minimum values. For example, with a projectile we might need to determine the maximum height reached. With an electrical circuit we might need to determine the condition for maximum power to be dissipated.

Consider a graph of y against x when the values of y depend in some way on the values of x. Points on the graph at which $dy/dx = 0$ are called *turning points* and can be:

- ***A local maximum***
  The term *local* is used because the value of y is only necessarily a maximum for points in the locality and there could be higher values of y elsewhere on the graph. Figure 4.10 shows such a maximum. At the maximum point A we have zero gradient for the tangent, i.e. $dy/dx = 0$. Consider two points P and Q close to A, with P having a value of x less than that at A and Q having a value greater than that at A. The gradient of the tangent at P is positive, the gradient of the tangent at Q is negative. Thus for a maximum we have the gradient changing from being positive prior to the turning point to negative after it.

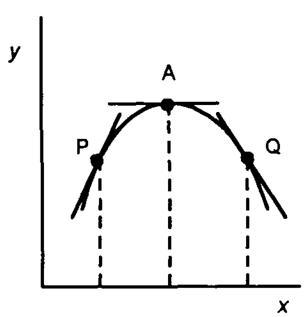


Figure 4.10 *A maximum*

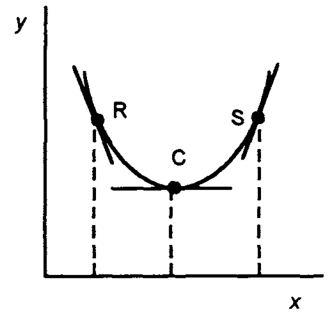


Figure 4.11 *A minimum*

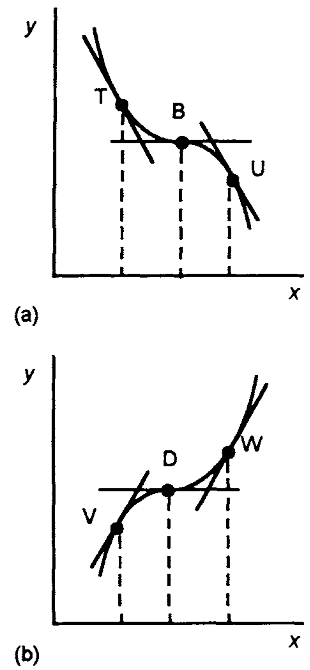


Figure 4.12 *Points of inflexion*

Key points

For the gradients in the vicinity of maxima, minima and points of inflexion, in moving from points before to after the turning point:

At a *maximum* the gradient changes from being positive to negative; the second derivative is negative.

At a *minimum* the gradient changes from being negative to positive; the second derivative is positive.

At a *point of inflexion* the sign of the gradient does not change.

- ***A local minimum***
  The term *local* is used because the value of y is only necessarily a minimum for points in the locality and there could be lower values of y elsewhere on the graph. Figure 4.11 shows such a minimum. At the minimum point C we have zero gradient for the tangent, i.e. $dy/dx = 0$. Consider two points R and S close to A, with R having a value of x less than that at C and S having a value greater than that at C. The gradient of the tangent at R is negative, the gradient of the tangent at S is positive. Thus for a minimum we have the gradient changing from being negative prior to the turning point to positive after it.

- ***A point of inflexion***
  Consider points of inflexion, as illustrated in Figure 4.12. At such points $dy/dx = 0$. However, in neither of the graphs is there a local maximum or minimum. In Figure 4.12(a), the gradient at a point T prior to the point is negative and the gradient at a point U after the point is also negative. In Figure 4.12(b), the gradient at a point V prior to the point is positive and the gradient at a point W after the point is also positive. For a point of inflexion the sign of the gradient prior to the point is the same as that after the point.

The gradient at a point on a graph is given by dy/dx. We can thus determine whether a turning point is a maximum, a minimum or a point of inflexion by considering how the value of dy/dx changes for a value of x smaller than the turning point value compared to that for a value of x greater than the turning point value.

There is an alternative method we can use to distinguish between maximum and minimum points. We need to establish how the gradient changes in going from points before to after turning points. Consider, for a maximum, a graph of the gradients plotted against x (Figure 4.13(a)). The gradients prior to the maximum are positive and decrease in value to become zero at the maximum. They then become negative and as x increases become more and more negative. The second derivative d^2y/dx^2 measures the rate of change of dy/dx with x, i.e. the gradient of the dy/dx graph. The gradient of the gradient graph is negative before, at and after the maximum point. Hence at a maximum d^2y/dx^2 is negative.

Consider a minimum (Figure 4.13(b)). The gradients prior to the minimum are negative and become less negative until they become zero at the minimum. As x increases beyond the minimum the gradients become positive, increasing in value as x increases. The second derivative d^2y/dx^2 measures the rate of change of dy/dx with x, i.e. the gradient of the dy/dx graph. The gradient of the gradient graph is positive before, at and after the minimum point. Hence at a minimum d^2y/dx^2 is positive.

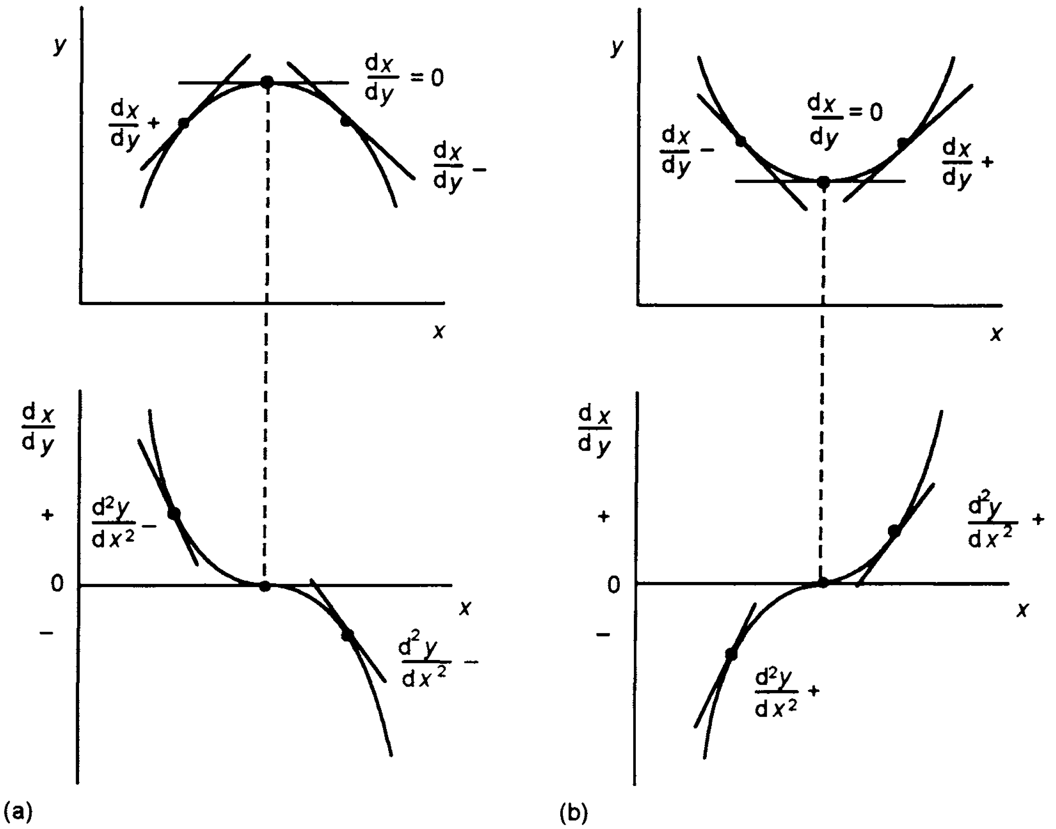


Figure 4.13 *(a) A maximum, (b) a minimum*

Example

Determine, and identify the form of, the turning points on a graph of the equation $y = 2x^3 - 3x^2 - 12x$.

Differentiating the equation gives

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

Thus the gradient of the graph is zero when $6x^2 - 6x - 12 = 0$. We can rewrite this as:

$$6(x^2 - x - 2) = 6(x + 1)(x - 2) = 0$$

The gradient is zero, and hence there are turning points, at $x = -1$ and $x = 2$.

To establish the form of these turning points consider the gradients just prior to and just after them.

Prior to the $x = -1$ turning point at $x = -2$, the gradient is $6x^2 - 6x - 12 = 6 \times (-2)^2 - 6 \times (-2) - 12 = 12$. After the point at $x = 0$ we have a gradient of -12. Thus the gradient prior to the $x = -1$ point is positive and after the point it is negative. The point is thus a maximum.

Consider the $x = 2$ turning point. Prior to the turning point at $x = 0$ the gradient is -12. After the turning point at $x = 3$ the gradient is $6 \times 3^2 - 6 \times 3 - 12 = 24$. Thus the gradient prior to the $x = 2$ point is negative and after the point it is positive. The point is thus a minimum.

Alternatively we could determine the form of the turning points by considering the sign of the second derivative at the points. The second derivative is obtained by differentiating the d*y*/d*x* equation. Thus

$$\frac{d^2y}{dx^2} = 12x - 6$$

At $x = -1$ then the second derivative is $12 \times (-1) - 6 = -18$. The negative value indicates that the point is a maximum. At $x = 2$ the second derivative is $12 \times 2 - 6 = 18$. The positive value indicates that the point is a maximum.

Figure 4.14 shows a graph of the equation $y = 2x^3 - 3x^2 - 12x$, showing the maximum at $x = -1$ and the minimum at $x = 2$.

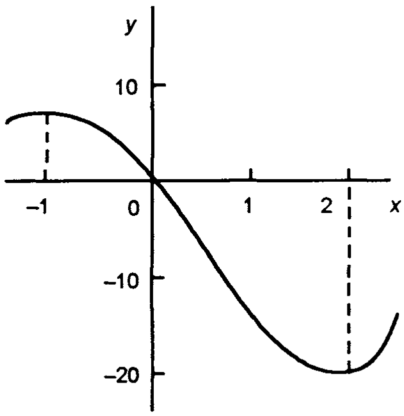


Figure 4.14 *Example*

Example

The displacement *y* in metres of an object is related to the time *t* in seconds by the equation $y = 5 + 4t - t^2$. Determine the maximum displacement.

Differentiating the equation gives:

$$\frac{dy}{dt} = 4 - 2t$$

d*y*/d*t* is 0 when $t = 2$ s. There is thus a turning point at the displacement $y = 5 + 4 \times 2 - 2^2 = 9$ m. We need to check that this is a maximum displacement. The gradient prior to the turning point at $t = 1$ has the value $4 - 2 = 2$. After the turning point at $t = 3$ it has the value $4 - 6 = -2$. The gradient changes from a positive value prior to the turning point to a negative value afterwards. It is thus a maximum.

Alternatively we could have established this by determining the second derivative. Differentiating $4 - 2t$ gives $d^2y/dx^2 = -2$. Thus, the turning point is a maximum.

Example

If the sum of two numbers is 40, determine the values which will give the minimum value for the sum of their squares.

Let the two numbers be x and y. Then we must have $x + y = 40$. We have to find the minimum value of S when we have $S = x^2 + y^2$. We need an equation which expresses the sum in terms of just one variable. Thus substituting from the previous equation gives

$$S = x^2 + (40 - x)^2 = x^2 + 1600 - 80x + x^2$$

$$= 2x^2 - 80x + 1600$$

Differentiating this equation, then

$$\frac{dS}{dx} = 4x - 80$$

The value of x to give a zero value for dS/dx is when $4x - 80 = 0$ and so when $x = 20$.

We can check that this is the value giving a minimum by considering the values of dS/dx prior to and after the point. Thus prior to the point at $x = 19$ we have $dS/dx = 4 \times 19 - 80 = -4$. After the point at $x = 21$ we have $dS/dx = 4 \times 21 - 80 = 4$. Thus dS/dx changes from being negative to positive. The turning point is thus a minimum. Alternatively we could check that this is a minimum by obtaining the second derivative. Differentiating $4x - 80$ gives $d^2S/dx^2 = 4$. Since this is positive then we have a minimum. Thus the two numbers which will give the required minimum are 20 and 20.

Example

Determine the maximum area of a rectangle with a perimeter of 32 cm.

If the width of the rectangle is w and its length L then the area $A = wL$. But the perimeter has a length of 32 cm. Thus $2w + 2L = 32$. If we eliminate w from the two equations:

$A = L(16 - L) = 16L - L^2$

Hence:

$$\frac{dA}{dL} = 16 - 2L$$

dA/dL is zero when $16 - 2L = 0$ and so when $L = 8$.

We can check that this gives a maximum area by considering values of the gradient at values of L below and above 8. At $L = 7$ the gradient is 2 and at $L = 9$ it is -2. It is thus a maximum. Alternatively we could have considered the second derivative. Since $d^2A/dL^2 = -2$ and so is negative, we have a maximum.

For a maximum area we must therefore have $L = 8$ cm, and, after substituting this value in $2w + 2L = 32$, $w = 8$ cm.

Maths in action

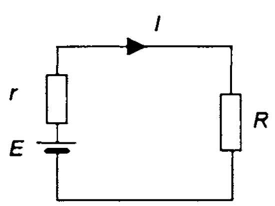


Figure 4.15 *Circuit*

Consider the circuit shown in Figure 4.15 where a d.c. source of e.m.f. E and internal resistance r supplies a load of resistance R. The power P supplied to the load is I^2R with the current I being $E/(R + r)$. Thus:

$$P = \frac{E^2R}{(R+r)^2}$$

Differentiating with respect to x by using the quotient rule gives:

$$\frac{dP}{dR} = \frac{(R+r)^2 - 2R(R+r)}{(R+r)^4} = \frac{E^2}{(R+r)^2} - \frac{2E^2R}{(R+r)^3}$$

$dP/dR = 0$ when:

$$\frac{2E^2R}{(R+r)^3} = \frac{E^2}{(R+r)^2}$$

and so $R = r$. We can check that this is the condition for maximum power transfer by considering the second derivative. We have the sum of two terms and so for the $E^2/(R + r)^2$ term, let $u = R + r$ and $y = E^2/u^2$. Then, $du/dR = 1$ and $dy/du = -2E^2/u^3$ and $dy/dR = -2E^2/(R + r)^3$. For $y = 2E^2R/(R + r)^3$ we can use the quotient rule to give $dy/dR = [(R + r)^3 2E^2 - 2E^2R3(R + r)^2]/(R + r)^6$ which can be simplified to $2E^2/(R + r)^3 - 6E^2R/(R + r)^4$. Hence:

$$\frac{d^2P}{dR^2} = -\frac{4E^2}{(R+r)^3} + \frac{6E^2R}{(R+r)^4}$$

With $R = r$ the second derivative is negative and so we have maximum power transfer.

4.1.5 Inverse functions

If we have a function y which is a continuous function of x then the derivative, i.e. the slope of the tangent to a graph of y plotted against x, is dy/dx. However, if we have x as a continuous function of y then the derivative, i.e. the slope of the tangent to a graph of x plotted against y, is dx/dy. How are these derivatives related? We might, for example, have $y = x^2$ and so $dy/dx = 2x$. For the inverse function $x = \sqrt{y}$ and $dx/dy = \frac{1}{2}y^{-1/2}$.

If we have a function $y = f(x)$ then we can write for the inverse $x = g(y)$. Thus $x = g\{f(x)\}$. Differentiating both sides of this equation with respect to x, using the chain rule for the right-hand side, gives:

$$1 = \frac{dx}{dy} \times \frac{dy}{dx}$$

Hence, with $y = f(x)$, the derivatives of the inverse function can be derived by using:

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \qquad [23]$$

For example, for $y = x^2$ we have $dy/dx = 2x$; the inverse function is $x = \sqrt{y}$ and $dx/dy = \frac{1}{2}y^{-1/2}$. Then $dy/dx = 1/(\frac{1}{2}y^{-1/2}) = 2\sqrt{y} = 2x$.

Example

Determine dy/dx for the function described by the equation $x = y^2 + 2y$.

It is easier to obtain dx/dy from the equation and thus the problem is tackled by doing that operation first. Thus $dx/dy = 2y + 2$. Then, using equation [23]:

$$\frac{dy}{dx} = \frac{1}{2y+2}$$

Logarithmic functions

Consider the function $y = \ln x$. We can write this as $x = e^y$. Differentiating x with respect to y gives:

$$\frac{dx}{dy} = e^y = x$$

Hence, using equation [23]:

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad [24]$$

Note that since x must be positive for $\ln x$ to have any meaning, equation [24] only applies for positive values of x.

> **Example**
>
> Determine the derivative of $y = e^{-x} \ln x$.
>
> Using the product rule then:
>
> $$\frac{dy}{dx} = e^{-x}\left(\frac{1}{x}\right) - \ln x\,(e^{-x}) = e^{-x}\left(\frac{1}{x} - \ln x\right)$$

Problems 4.1

1 Determine the derivatives of the following functions:

(a) $y = x^5$, (b) $y = 2x^{-4}$, (c) $y = -3x^2$, (d) $y = \frac{1}{2}x$,

(e) $y = 2\pi x^2$, (f) $y = \tan 3x$, (g) $y = 5 \cos 2x$,

(h) $y = 4\,e^{x/2}$, (i) $y = 2\,e^{-2x}$, (j) $y = 3\,e^{3x}$, (k) $y = \frac{5}{3\sqrt{x}}$,

(l) $y = \frac{6}{5x^2}$, (m) $y = \frac{7}{\sqrt{3x}}$, (n) $y = \frac{5}{(2x)^3}$, (o) $y = \sqrt{3x}$,

(p) $y = (3x - 2x^2)(5 + 4x)$, (q) $y = 5x \sin x$, (r) $x\,e^{x/2}$,

(s) $(x^2 + 1) \sin x$, (t) $y = \frac{2x+1}{x-6}$, (u) $y = \frac{x+1}{\sqrt{x}}$,

(v) $y = \frac{\sin x}{x}$, (w) $y = \frac{e^{2x}}{x^2+1}$, (x) $y = \frac{\sinh 2x}{\cosh 3x}$,

(y) $y = \frac{1}{2-7x}$, (z) $y = \frac{1}{\sqrt{3-x}}$

2 Determine the second derivatives of the following functions:

(a) $y = x^2 + 2x$, (b) $y = \sin 2x$, (c) $y = \frac{1}{x^2}$,

(d) $y = 3x^4 - x^2 - \frac{1}{x}$, (e) $y = x^4 + 2x^3 - 8x + 5$,

(f) $y = \frac{x+1}{\sqrt{x}}$

3 Determine the velocity and acceleration after a time of 2 s for an object which has a displacement x which is a function of time t and given by $x = 12 + 15t - 2t^2$, with t being in seconds.

4 Determine the velocity and acceleration at a time t for an object which has a displacement x in metres given by $x = 3 \sin 2t + 3 \cos 3t$, t being in seconds.

5 The voltage v, in volts, across a capacitor of capacitance 2 μF varies with time t, in seconds, according to the equation $v = 3 \sin 5t$. Determine how the current varies with time.

6 The current i, in amps, through an inductor of inductance 0.05 H varies with time t, in seconds, according to the equation $i = 10(1 - e^{-100t})$. Determine how the potential difference across the inductor varies with time.

7 The volume of a cone is one-third the product of the base area and the height. For a cone with a height equal to the base radius, determine the rate of change of cone volume with respect to the base radius.

8 The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$. Determine the rate of change of the volume with respect to the radius.

9 With the Doppler effect, the frequency f_o heard by an observer when a sound source of frequency f_s is moving away from the observer with a velocity v is given by $f_o = f_s/(1 + v/c)$, where c is the velocity of sound. Determine the rate of change of the observed frequency with respect to the velocity.

10 The length L of a metal rod is a function of temperature T and is given by the equation $L = L_0(1 + aT + bT^2)$. Determine an equation for the rate of change of length with temperature.

11 Determine and identify the form of the turning points on graphs of the following functions:

(a) $y = x^2 - 4x + 3$, (b) $y = x^3 - 6x^2 + 9x + 3$,

(c) $y = x^5 - 5x$, (d) $y = \sin x$ for x between 0 and 2π,

(e) $y = 2x^3 + 3x^2 - 12x + 3$

12 A cylindrical container, open at one end, has a height of h m and a base radius of r m. The total surface area of the container is to be 3π m². Determine the values of h and r which will make the volume a maximum.

13 A cylindrical metal container, open at one end, has a height of h cm and a base radius of r cm. It is to have an internal volume of 64π cm³. Determine the dimensions of the container which will require the minimum area of metal sheet in its construction.

14 The bending moment M of a uniform beam of length L at a distance x from one end is given by $M = \frac{1}{2}wLx - \frac{1}{2}wx^2$, where w is the weight per unit length of beam. Determine the value of x at which the bending moment is a maximum.

15 The deflection y of a beam of length L at a distance x from one end is found to be given by $y = 2x^4 - 5Lx^3 + 2L^2x^2$. Determine the values of x at which the deflection is a maximum.

16 Determine the maximum value of the alternating voltage described by the equation $v = 40 \cos 1000t + 15 \sin 1000t$ V.

17 The intensity of illumination from a point light source of intensity I at a distance d from it is I/d^2. Determine the point along the line between two sources 10 m apart at which the intensity of illumination is a minimum if one of the sources has eight times the intensity of the other.

18 Determine the maximum rate of change with time of the alternating current $i = 10 \sin 1000t$ mA, the time t being in seconds.

19 The deflection y of a propped cantilever of length L at a distance x from the fixed end is given by:

$$y = \frac{1}{EI}\left(\frac{5wLx^3}{48} - \frac{wL^2x^2}{16} - \frac{wx^4}{24}\right)$$

where w is the weight per unit length and E and I are constants. Determine the value of x at which the deflection is a maximum.

20 The e.m.f. E produced by a thermocouple depends on the temperature T and is given by $E = aT + bT^2$. Determine the temperature at which the e.m.f. is a maximum.

21 The horizontal range R of a projectile projected with a velocity v at an angle θ to the horizontal is given by $R = (v^2/g) \sin 2\theta$. Determine the angle at which the range is a maximum for a particular velocity.

22 A 100 cm length of wire is to be bent to form two squares, one with side x and the other with side y. Determine the values of x and y which give the minimum area enclosed by the squares.

23 The rate r at which a chemical reaction proceeds depends on the quantity x of a chemical and is given by $r = k(a - x)(b + x)$. Determine the maximum rate.

24 A cylinder has a radius r and height h with the sum of the radius and height being 2 m. Determine the radius giving the maximum volume.

25 A rectangle is to have an area of 36 cm². Determine the lengths of the sides which will give a minimum value for the perimeter.

26 An open tank is to be constructed with a square base and vertical sides and to be able to hold, when full to the brim, 32 m³ of water. Determine the dimensions of the tank if the area of sheet metal used is to be a minimum.

4.2 Integration

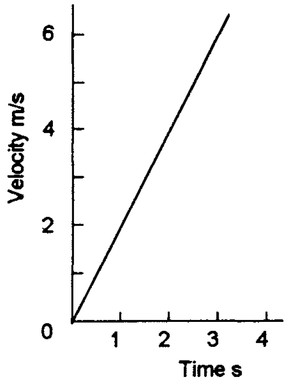


Figure 4.14 *Velocity–time graph*

Key point

Differentiation is the determination of the relationship for the gradient of a graph. We can define integration as the mathematical process which reverses differentiation, i.e. given the gradient relationship then finding the equation which was responsible for it.

Integration can be considered to be the mathematical process which is the reverse of the process of differentiation. It also turns out to be a process for finding areas under graphs.

As an illustration of the application of integration in engineering as the reverse of differentiation, consider the situation where the velocity v of an object varies with time t, say $v = 2t$ (Figure 4.14). Since velocity is the rate of change of distance x with time we can write this as:

$$\frac{dx}{dt} = 2t$$

Thus we know how the gradient of the distance–time graph varies with time. Integration is the method we can use to determine from this how the distance varies with time. We thus start out with the gradients and find the distance–time graph responsible for them, the reverse of the process used with differentiation.

4.2.1 Integration as the reverse of differentiation

Suppose we have an equation $y = x^2$. When this equation is differentiated we obtain the derivative of $dy/dx = 2x$. Thus, in this case, when given the gradient as $2x$ we need to find the equation which on being differentiated gave $2x$. Thus, integrating $2x$ should give us x^2. However, the derivative of $x^2 + 1$ is also $2x$, likewise the derivative of $x^2 + 2$, the derivative of $x^2 + 3$, and so on. Figure 4.15 shows part of the family of graphs which all have the gradients given by $2x$.

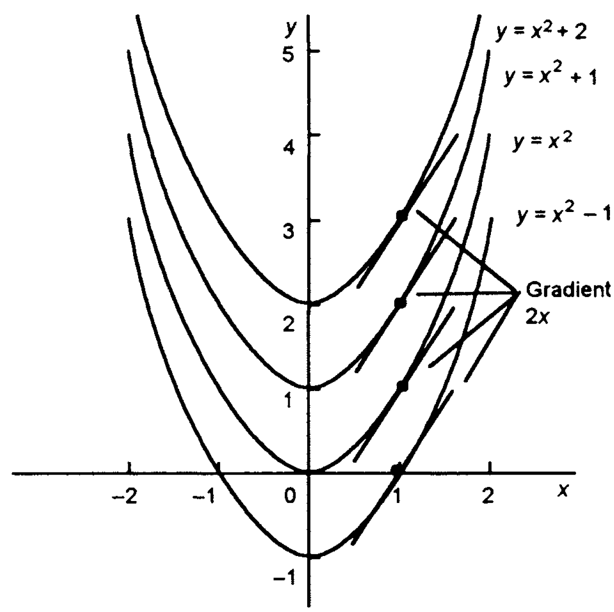


Figure 4.15 *dy/dx = 2x. All the above graphs have gradients which are 2x.*

Thus, for each of the graphs, at a particular value of x, such as $x = 1$, they all give the same gradient of 2. Thus in the integration of $2x$ we are not sure whether there is a constant term or not, or what value it might have. Hence a constant C has to be added to the result. Thus the outcome of the integration of $2x$ has to be written as being $x^2 + C$. The integral, which has to have a constant added to it, is referred to as an *indefinite integral.*

To indicate the process of integration a special symbol:

$$\int f(x)\, dx \tag{25}$$

is used. This sign indicates that integration is to be carried out and the dx that x is the variable we are integrating with respect to. Thus the integration referred to above can be written as

$$\int 2x\, dx = x^2 + C$$

Integrals of common functions

The integrals of functions can be determined by considering what equation will give the function when differentiated. For example, consider:

$$\int x^n\, dx$$

Considering integration as the inverse of differentiation, the question becomes as to what function gives x^n when differentiated. The derivative of x^{n+1} is $(n + 1)x^n$. Thus, we have the derivative of $x^{n+1}/(n + 1)$ as x^n. Hence:

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C \tag{26}$$

This is true for positive, negative and fractional values of n other than $n = -1$, i.e. the integral of x^{-1}. For the integral of x^{-1}, i.e.

$$\int \frac{1}{x}\, dx$$

then since the derivative of $\ln x$ is $1/x$:

$$\int \frac{1}{x}\, dx = \ln x + C \tag{27}$$

This only applies if x is positive, i.e. $x > 0$. If x is negative, i.e. $x < 0$, then the integral of $1/x$ in such a situation is *not* $\ln x$. This is because we cannot have the logarithm of a negative number as a real quantity. To show that only positive values of a quantity are to be considered, we write it as $|x|$.

Consider the integral of the exponential function e^x, i.e.

$$\int e^x\, dx$$

The derivative of e^x is e^x. Thus:

Key points

Function being integrated	Outcome of the integration
$\int ax^n \mathrm{d}x$	$\frac{ax^n}{n+1}+C$, n not -1
$\int a\,\mathrm{d}x$	$ax+C$
$\int \frac{1}{x}\mathrm{d}x$	$\ln x+C$
$\int e^{ax}\mathrm{d}x$	$\frac{1}{a}e^{ax}+C$
$\int \sin ax\mathrm{d}x$	$-\frac{1}{a}\cos ax+C$
$\int \cos ax\mathrm{d}x$	$\frac{1}{a}\sin ax+C$

$$\int e^x\,\mathrm{d}x = e^x + C \qquad [28]$$

The key points shows some functions and their integrals.

Example

Evaluate the integrals:

(a) $\int x^4\mathrm{d}x$, (b) $\int x^{1/2}\mathrm{d}x$, (c) $\int x^{-4}\mathrm{d}x$, (d) $\int x^{-1}\mathrm{d}x$,

(e) $\int \cos 4x\mathrm{d}x$, (f) $\int e^{2x}\mathrm{d}x$, (g) $\int 5\mathrm{d}x$.

(a) Using $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1}+C$:

$$\int x^4\mathrm{d}x = \frac{x^{4+1}}{4+1}+C = \frac{x^5}{5}+C$$

(b) Using $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1}+C$:

$$\int x^{1/2}\mathrm{d}x = \frac{x^{1/2+1}}{1/2+1}+C = \frac{x^{3/2}}{3/2}+C = \frac{2}{3}x^{3/2}+C$$

(c) Using $\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1}+C$:

$$\int x^{-4}\mathrm{d}x = \frac{x^{-4+1}}{-4+1}+C = \frac{x^{-3}}{-3}+C = -\frac{1}{3}x^{-3}+C$$

(d) Using the relationship giving in the table:

$$\int x^{-1}\mathrm{d}x = \ln x + C$$

(e) Using $\int \cos ax\mathrm{d}x = \frac{1}{a}\sin x + C$:

$$\int \cos 4x\mathrm{d}x = \frac{1}{4}\sin 4x + C$$

(f) Using $\int e^{ax}\mathrm{d}x = \frac{1}{a}e^x + C$:

$$\int e^{2x}\mathrm{d}x = \frac{1}{2}e^{2x}+C$$

(g) Using $\int a\,\mathrm{d}x = ax + C$:

$$\int 5\mathrm{d}x = \int 5x^0\mathrm{d}x = 5x + C$$

The above is just a particular version of the standard integral:

$$\int x^n\mathrm{d}x = \frac{x^{n+1}}{n+1}+C$$

with $n = 0$ since $x^0 = 1$.

Integral of a sum

Key point

The integral of a sum of two functions equals the sum of their individual integrals.

The derivative of, for example, $x^2 + x$ is the derivative of x^2 plus the derivative of x, i.e. it is $2x + 1$. The integral of $2x + 1$ is thus $x^2 + x + C$. Thus, the integral of the sum of a number of functions is the sum of their separate integrals.

Example

Determine the integral $\int(x^3+2x+4)\,dx$.

We can write this as:

$$\int(x^3+2x+4)\,dx = \int x^3\,dx + \int 2x\,dx + \int 4\,dx$$

Hence the integral is:

$$\frac{x^4}{4}+P+x^2+Q+4x+R$$

where P, Q and R are constants. We can combine these constants into a single constant C. Hence the integral is:

$$\frac{x^4}{4}+x^2+4x+C$$

Finding the constant of integration

The solution given by the above integration is a general solution and includes a constant. As was indicated earlier in Figure 4.15 the integration of $2x$ gives $y = x^2 + C$. This solution indicates a family of possible equations which could give $dy/dx = 2x$. We can, however, find a *particular solution* if we are supplied with information giving specific coordinate values which have to fall on the graph curve. Thus, in this case, we might be given the condition that when $y = 1$ we have $x = 1$. This must fit the equation $y = x^2 + C$ and can only be the case when $C = 0$. Hence the solution is $y = x^2$.

Key point

Note that when we integrate a relationship for dy/dx with respect to x we obtain a relationship for y; when we integrate a relationship for d^2y/dx^2 with repect to x we obtain a relationship for dy/dx.

$$\int \frac{dy}{dx}\,dx = y + C$$

$$\int \frac{d^2y}{dx^2}\,dx = \frac{dy}{dx} + C$$

Example

Determine the equation of a graph if it has to have y = 0 when $x = 2$ and has a gradient given by:

$$\frac{dy}{dx} = 3x+2$$

To obtain the general solution, i.e. the family of curves which fit the above gradient equation, we integrate. Thus:

$$y = \int(3x+2)\,dx = \tfrac{3}{2}x^2+2x+C$$

The particular curve we require must have $y = 0$ when $x = 2$. Putting this data into the equation gives $2 = 0 + 0 + C$. Hence $C = 2$ and so the particular solution is:

$$y = \tfrac{3}{2}x^2 + 2x + 2$$

Example

A curve is such that its gradient is described by the equation $dy/d\theta = \cos\theta$ and $y = 1$ when $\theta = \pi/2$ radians. Find the equation of the curve.

Here we have the relationship $dy/d\theta = \cos\theta$, and so integration gives:

$$y = \int \cos\theta \, d\theta = \sin\theta + C$$

This is the general solution. To find the specific equation we need to evaluate C by substituting the known conditions, namely that $y = 1$ when $\theta = \pi/2$ radians. Thus:

$$1 = \sin(\pi/2) + C$$

$$1 = 1 + C$$

Therefore $C = 0$ and the required equation is:

$$y = \sin\theta$$

Example

At any point on a curve we have a gradient of $dy/dt = 3 \sin t$. Find the equation of the curve given that $y = 2$ when t has the value of 25°.

Given the relationship $dy/dt = 3 \sin t$, integration gives:

$$y = \int 3\sin t \, dt = 3\int \sin t \, dt = 3(-\cos t) + C$$

The general equation is thus $y = -3 \cos t + C$. But $y = 2$ when $t = 25°$ and so:

$$2 = -3\cos 25° + C = -3(0.91) + C$$

Thus $C = 4.73$ and the specific equation, which is the required equation, is:

$$y = 4.73 - 3\cos t.$$

Maths in action

The bending of beams

See the Maths in action in Section 4.1.3 for a preliminary discussion of the bending of beams.

The deflection y of a beam can be obtained by integrating the differential equation:

$$\frac{d^2y}{dx^2} = -\frac{M}{EI}$$

with respect to x to give:

$$\frac{dy}{dx} = -\frac{1}{EI}\int M\,dx + A$$

with A being the constant of integration and then carrying out a further integration with respect to x to give:

$$y = -\frac{1}{EI}\int\left[\int M\,dx + A\right] + B = -\frac{1}{EI}\int\int M\,dx + Ax + B$$

with B being a constant of integration.

At the point where maximum deflection occurs, the slope of the deflection curve will be zero and thus the point of maximum deflection can be determined by equating dy/dx to zero.

As an illustration, consider a horizontal cantilever supporting a load at its free end (Figure 4.16). The bending moment a distance x from the fixed end is given by $M = -F(L - x)$ and so the differential equation becomes:

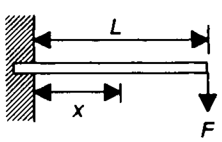


Figure 4.16 *Example*

$$EI\frac{d^2y}{dx^2} = -M = F(L - x)$$

Integrating with respect to x gives:

$$EI\frac{dy}{dx} = FLx - \frac{Fx^2}{2} + A$$

Since the slope of the beam $dy/dx = 0$ at the fixed end where $x = 0$, then we must have $A = 0$ and so:

$$EI\frac{dy}{dx} = FLx - \frac{Fx^2}{2}$$

Integrating again gives:

$$EIy = \frac{FLx^2}{2} - \frac{Fx^3}{6} + B$$

Since, at the fixed end we have zero deflection, i.e. we have $y = 0$ at $x = 0$, then we must have $B = 0$ and so:

$$EIy = \frac{FLx^2}{2} - \frac{Fx^3}{6} = \frac{Fx^2}{6}(3L - x)$$

When $x = L$:

$$y = \frac{FL^3}{3EI}$$

For beams with a number of concentrated loads, there will be discontinuities in the bending moment diagram and so we cannot write a single bending moment equation to cover the entire beam but have to write separate equations for each part of the beam between adjacent loads. Integration of each expression then gives the deflections relationship for each part of the beam. There is an alternative and that involves writing a single equation using, what are termed, Macaulay's brackets. For a discussion and examples of this method, see the companion book: *Mechanical Engineering Systems* by R. Gentle, P. Edwards and W. Bolton.

4.2.2 Integration as the area under a graph

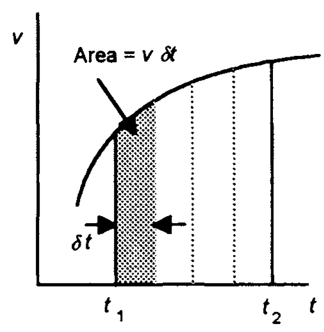


Figure 4.17 *Velocity-time graph*

Consider a moving object and its graph of velocity v against time t (Figure 4.17). The distance travelled between times of t_1 and t_2 is the area under the graph between those times. If we divide the area into a number of equal width strips then we can represent this area under the velocity–time graph as being the sum of the areas of these equal width strip areas, as illustrated in Figure 4.17. If t is the value of the time at the centre of a strip of width δt and v the velocity at this time, then a strip has an area of $v\,\delta t$. Thus the area under the graph between the times t_1 and t_2 is equal to the sum of the areas of all such strips between the times t_1 and t_2,

distance travelled = sum of the areas of all the strips between t_1 and t_2

We can write this summation as:

$$x = \sum_{t=t_1}^{t=t_2} v\,\delta t$$

Key point

The sigma notation used for a summation has the numbers above it and below indicating where the summation is to start and finish.

The Σ sign is used to indicate that we are carrying out a summation of a number of terms. The limits between which this summation is to be carried out are indicated by the information given below and above the sign. If we make δt very small, i.e. let δt tend to 0, then we denote it by dt. The sum is then the sum of a series of very narrow strips and is written as:

$$x = \lim_{\delta t \to 0} \sum_{t=t_1}^{t=t_2} v\,\delta t = \int_{t_1}^{t_2} v\,\mathrm{d}t \qquad [29]$$

The integral sign is an "S" for summation and the t_1 and t_2 are said to be the limits of the range of the variable t. Here x is the *integral* of the v with time t between the limits t_1 and t_2. The process of obtaining x in this way is termed *integration*. Because the integration is between specific limits it is referred to as a *definite integral*.

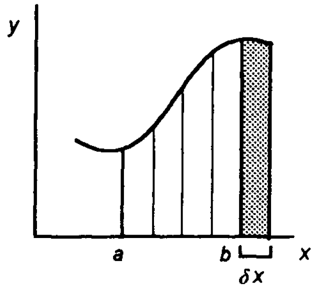


Figure 4.18 *Area increased by one strip*

Integration as reverse of differentiation and area under a graph

The definitions of integration in terms of the reverse of differentiation and as the area under a graph describe the same concept. Suppose we increase the area A under a graph of y plotted against x by one strip (Figure 4.18). Then the increase in the area δA is the area of this strip. Thus:

increase in area $\delta A = y\,\delta x$

So we can write:

$$\frac{\delta A}{\delta x} = y$$

In the limit as δx tends to 0 then we can write dA/dx and so

$$\frac{\mathrm{d}A}{\mathrm{d}x} = y$$

With integration defined as the inverse of differentiation then the integration of the above equation gives the area A, i.e.

$$A = \int y\,\mathrm{d}x \qquad [30]$$

This is an indefinite integral, which is the same as that given by the definition for integration as the area under a graph when limits are imposed. An *indefinite integral* has no limits and the result has a constant of integration. Integration between specific limits gives a *definite integral*.

Key points

The notation

$\int f(x)\,\mathrm{d}x$

with no limits specified stands for any antiderivative of $f(x)$. When we evaluate such an integral there will be a constant in the answer.

$\int_a^b f(x)\,\mathrm{d}x$

is termed a definite integral because it takes a definite value, representing an area under a curve. When we evaluate such an integral there is no constant in the answer. The integral gives the area under the curve of the function f(x) between x = a and x = b and:

$\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = F(b) - F(a)$

The term inside the square brackets is the result of the integration before the limits have been applied.

Areas under graphs

Consider the integration of y with respect to x when we have $y = 2x$. This has no specified limits and so is an indefinite integral,

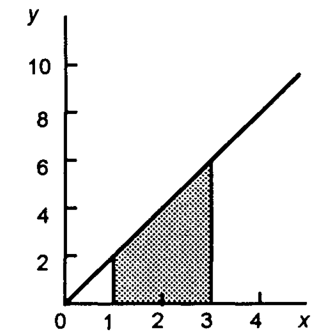


Figure 4.19 $y = 2x$

with the solution as the function which differentiated would give $2x$:

$$\int 2x\,dx = x^2 + C$$

Now consider the area under the graph of $y = 2x$ between the limits of $x = 1$ and $x = 3$ (Figure 4.19). We can write this as the definite integral:

$$\int_1^3 2x\,dx$$

$$\int_1^3 2x\,dx = [x^2 + C]_1^3$$

The square brackets round the $x^2 + C$ are used to indicate that we have to impose the limits of 3 and 1 on it. Thus the integral is the value of $x^2 + C$ when $x = 3$ minus the value of $x^2 + C$ when $x = 1$.

$$\int_1^3 2x\,dx = (9 + C) - (1 + C) = 8 \text{ square units}$$

The constant term C vanishes when we have a definite integral.

Note that an area below the x-axis is negative. If the area is required when part of it is below the x-axis then the parts below and above the x-axis must be found separately and then added, disregarding the sign of the area.

Example

Determine the area between a graph of $y = x + 1$ and the x-axis between $x = -2$ and $x = 4$.

Figure 4.20 shows the graph. The area required is that between the values of x of -2 and 4. We can break this area down into a number of elements. The area under the graph between $x = 0$ and $x = 4$ is that of a rectangle 4×1 plus a triangle ½(4 × 4) and so is +12 square units. The area between $x = -1$ and 0 is that of a triangle ½(1 × 1) = 0.5 square units and the area between $x = -1$ and $x = -2$ is a triangular area below the axis and so is negative and given by ½(1 × 1) = −0.5 square units. Hence the total area under the graph is +12 + 0.5 − 0.5 = 12 square units.

Alternatively we can consider this area as the integral:

$$\text{area} = \int_{-2}^{4} (x+1)\,dx = \left[\frac{x^2}{2} + x + C\right]_{-2}^{4}$$

and so:

$$\text{area} = (8 + 4 + C) - (2 - 2 + C) = 12 \text{ square units}$$

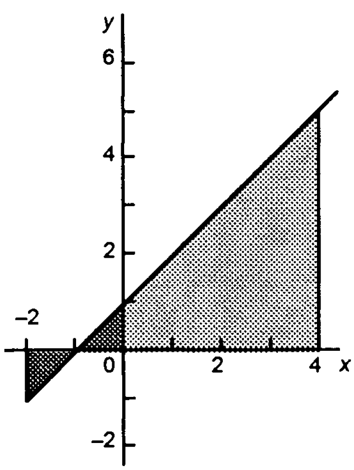


Figure 4.20 *Example*

Example

Determine the value of the integral $\int_{-2}^{4} e^{2x}\,dx$.

We can consider that this integral represents the area under the graph between $x = -2$ and $x = 4$ of e^{2x} plotted against x.

$$\int_{-2}^{4} e^{2x}\,dx = \left[\tfrac{1}{2}e^{2x} + C\right]_{-2}^{4}$$

$$= \tfrac{1}{2}e^{8} - \tfrac{1}{2}e^{-4} = 1490.479 - 0.009$$

The value of the integral is thus 1490.470.

Example

Determine the value of the integral $\int_{0}^{\pi/3} \cos 2x\,dx$.

We can consider that this integral represents the area under the graph between $x = 0$ and $x = \pi/3$ of $\cos 2x$ plotted against x.

$$\int_{0}^{\pi/3} \cos 2x\,dx = \left[\tfrac{1}{2}\sin 2x + C\right]_{0}^{\pi/3}$$

$$= \tfrac{1}{2}\sin 2\pi/3 - \tfrac{1}{2}\sin 0 = 0.433$$

The value of the integral is thus 0.433.

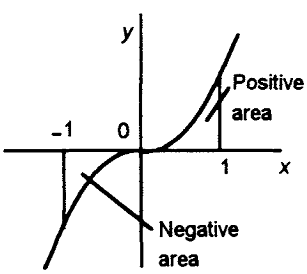


Figure 4.21 *Example*

Example

Find the areas under the curve $y = x^3$ between (a) $x = 0$ and $x = 1$, (b) $x = -1$ and $x = 1$.

(a) Figure 4.21 shows the graph. The area is:

$$\int_{0}^{1} x^3 = \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4}$$

(b) The area taking into account the sign of y, is:

$$\int_{-1}^{1} x^3 = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$$

The area is zero because the area between $x = -1$ and $x = 0$ is negative. What we have is the sum of two areas:

$$\int_{-1}^{1} x^3 = \int_{-1}^{0} x^3\,dx + \int_{0}^{1} x^3\,d$$

$$\int_{-1}^{0} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{0} = 0 - \frac{1}{4} = -\frac{1}{4}$$

$$\int_{0}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{4} - 0 = \frac{1}{4}$$

The sum of the two areas is thus zero. If we want the total area, regardless of sign, between the axis and the curve for any function then we have to determine the positive and negative elements separately and then, ignoring the sign, add them. For this curve this gives ½.

Maths in action

Work

With a constant force F acting on a body and a displacement x in the direction of the force, then the work done W on a body, i.e. the energy transferred to it, is given by $W = Fx$.

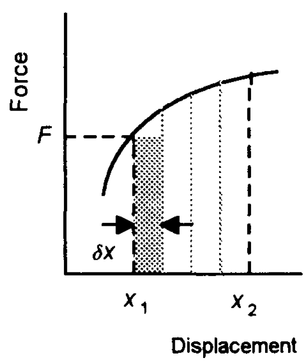


Figure 4.22 *Force-displacement graph*

Consider a variable force described by the graph shown in Figure 4.22 for the force applied to an object and how it varies with the displacement of that object. For a small displacement δx we can consider the force to be effectively constant at F. Thus the work done for that displacement is $F\,\delta x$. This is the area of the strip under the force–distance graph. If we want the work done in changing the displacement from x_1 to x_2 then we need to determine the sum of all such strips between these displacements, i.e. the total area under the graph between the ordinates for x_1 and x_2. Thus:

$$\text{work done} = \sum_{x=x_1}^{x=x_2} F\,\delta x$$

If we make the strips tend towards zero thickness then the above summation becomes the integral, i.e.

$$\text{work done} = \int_{x_1}^{x_2} F\,dx$$

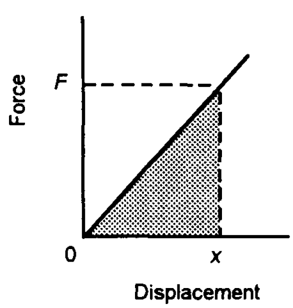


Figure 4.23 *Stretching a spring*

Consider the work done in stretching a spring when a force F is applied and causes a displacement change in its point of application, i.e. an extension, from 0 to x if $F = kx$, where k is a constant. Figure 4.23 shows the force–distance graph.

The work done is the area under the graph between 0 and x. This is the area of a triangle and so the work done is $\frac{1}{2}Fx$. Since $F = kx$ we can write this as $\frac{1}{2}kx^2$.

We could have solved this problem by integration. Thus:

$$\text{work done} = \int_0^x F\,dx = \int_0^x kx\,dx = \left[\tfrac{1}{2}kx^2 + C\right]_0^x = \tfrac{1}{2}kx^2$$

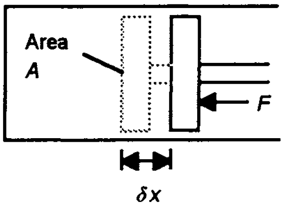


Figure 4.24 *Compressing a gas*

As a further illustration, consider the work done as a result of a piston in Figure 4.24 being moved to reduce the volume of a gas. The work done in moving the piston through a small distance δx when the force is F is $F\,\delta x$. Since pressure is force per unit area, then if the force acts over an area A the pressure $p = F/A$. Thus:

$$\text{work done} = F\,\delta x = pA\,\delta x$$

But $A\,\delta x$ is the change in volume δV of the gas. Hence, the work done = $p\,\delta V$. The total work done in changing the volume of a gas from V_1 to V_2 is thus:

$$\text{work done} = \sum_{V_1}^{V_2} p\delta V$$

If we consider δV tending to zero then we can write

$$\text{work done} = \int_{V_1}^{V_2} p\,dV$$

For a gas that obeys Boyle's law, i.e. pV = a constant k, the work done in compressing a gas from a volume V_1 to V_2 is thus:

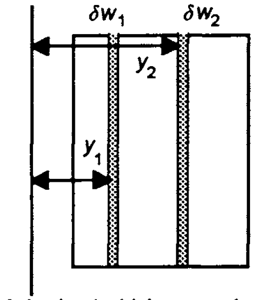
$$\text{work done} = \int_{V_1}^{V_2} p\,dV = \int_{V_1}^{V_2} \frac{k}{V}\,dV = [\ln V + C]_{V_1}^{V_2}$$

Hence the work done is $\ln V_2 - \ln V_1$.

Maths in action

Centre of gravity and centroid

The weight of a body is made up of the weights of each constituent particle, each such particle having its weight acting at a different point. However, it is possible to replace all the weight forces of an object by a single weight force acting at a particular point, this point being termed the *centre of gravity*.



Axis about which moments are taken

Figure 4.25 *Moments of elements*

If we consider a sheet to be made of a large number of small strip elements of mass at different distances from an axis (Figure 4.25) then the weight of each element will give rise to a moment about that axis. Thus the total moment due to all the weight elements is $\delta w_1 x_1 + \delta w_2 x_2 + \delta w_3 x_3 + \dots$. If a single weight W at a distance $\bar{x}$ is to give the same moment, then:

$W\bar{x} = \sum \delta w\, x$ for all the strips in the sheet

Thus the distance of the centre of gravity from the chosen axis is:

$$\bar{x} = \frac{\sum_1^n \delta w_i x_i}{W}$$

For a thin flat plate of uniform density, the weight of an element is proportional to its area. We then refer to the *centroid* since it is purely geometric. The distance of the centroid from the chosen axis is thus:

$$\bar{x} = \frac{\sum_1^n \delta a_i x_i}{A}$$

where δa represents the area of an elemental strip. The product of an area and its distance from an axis is known as the *first moment of area* of that area about the axis. Thus the centroid distance from an axis is the sum of the first moments of all the area elements divided by the sum of all the areas of the elements.

If we consider infinitesimally small elements, i.e. $\delta a \to 0$, then we can write:

$$\bar{x} = \frac{\int x\, \mathrm{d}a}{A}$$

Consider the determination of the centroid of a triangular area (Figure 4.26). Consider a small strip of area $\delta A = x\,\delta y$. By similar triangles $x/(h-y) = b/h$ and so $x\,\delta y = [b(h-y)/h]\,\delta y$. The total area $A = \tfrac{1}{2}bh$. Hence, the y coordinate of the centroid is:

$$\bar{y} = \frac{2}{bh}\int_0^h y\frac{b(h-y)}{h}\,\mathrm{d}y = \tfrac{1}{3}h$$

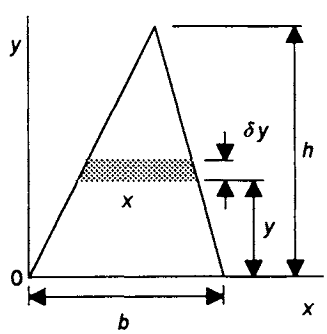


Figure 4.26 *Triangular area*

The centroid is located at one-third the altitude of the triangle. The same result is obtained if we consider the location with respect to the other sides. The centroid is at one-third the altitude along of the lines drawn from each apex to the opposite side.

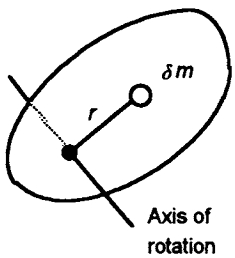


Figure 4.27 *Rotation of a rigid body*

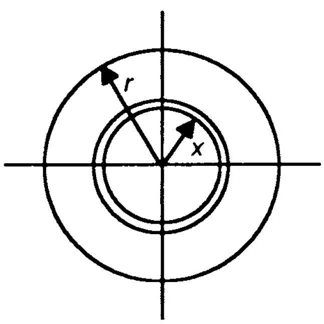


Figure 4.28 *Disc*

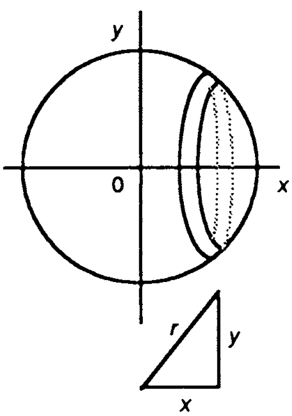


Figure 4.29 *Sphere*

Maths in action

Moment of inertia

Consider a rigid body rotating with a constant angular acceleration α about some axis (Figure 4.27). We can consider the body to be made up of small elements of mass δm. For such an element a distance r from the axis of rotation we have a linear acceleration of $a = r\alpha$. Thus the force acting on the element is $\delta m \times r\alpha$. The moment of this force is thus $Fr = r^2\alpha\,\delta m$. The total moment, i.e. torque T, due to all the elements of mass in the body is thus:

$$T = \sum r^2 \alpha\,\delta m \text{ for all the elements}$$

Thus if we have elements of mass at radial distance from 0 to R, in the limit as $\delta m \to 0$:

$$T = \int_0^R r^2 \alpha\,\mathrm{d}m$$

Since α is a constant we can write the above equation as:

$$T = \left(\int_0^R r^2\,\mathrm{d}m\right)\alpha = I\alpha$$

where *I is the moment of inertia*

As an illustration consider the determination of the moment of inertia of a uniform disc about an axis through its centre and at right angles to its plane. Figure 4.28 shows the disc with an element of mass being chosen as a disc with a radius x and width δx. The element is a strip of length $2\pi x$ and so an area of $2\pi x\,\delta x$. If the mass of the disc is m per unit area, then the mass of the element is $\delta m = 2\pi mx\,\delta x$. The moment of inertia of the element is $x^2\,\delta m = 2\pi mx^3\,\delta x$. Thus the moment of inertia of the disc is:

$$I = \int_0^r 2\pi mx^3\,\mathrm{d}x = 2\pi m\left[\frac{x^4}{4}\right]_0^r = \tfrac{1}{2}\pi mr^4$$

As another illustration, consider a sphere of radius r and mass per unit volume m. If we take a thin slice of thickness δx of the sphere perpendicular to the diameter about which the moment of inertia is to be determined and a distance x from the sphere centre (Figure 4.29), then with the slice radius y we have an element of volume $\pi y^2\,\delta x$ and hence mass $\pi my^2\,\delta x$. The moment of inertia of a disc is ½ mass × radius² (see the previous example) and thus the moment of inertia of the slice is $\tfrac{1}{2}(\pi my^2\,\delta x)y^2$ and the moment of inertia of the sphere as the sum of all the slices as $\delta x \to 0$ is:

$I = \int_{-r}^{r} \frac{1}{2}\pi m y^4 \, dx$

Since $r^2 = y^2 + x^2$:

$I = \frac{1}{2}\pi m \int_{-r}^{r} (r^2 - x^2)^2 \, dx = \frac{1}{2}\pi m \int_{-r}^{r} (r^4 - 2r^2x^2 + x^4) \, dx$

$= \frac{1}{2}\pi m \left[r^4 x - \frac{2}{3}r^2x^3 + \frac{1}{5}x^5 \right]_{-r}^{r} = \frac{8}{15}\pi m r^5$

The total mass M of the sphere is $\frac{4}{3}\pi m r^3$ so $I = \frac{2}{5}Mr^2$.

Maths in action

Second moment of area

Consider a beam that has been bent into the arc of a circle so that the uppermost surface is in tension and the lower surface in compression (Figure 4.30). The upper surface has increased in length and the lower surface decreased in length; between the two there is a plane which is unchanged in length; this is called the neutral plane and the line where the plane cuts the cross-section of the beam is the neutral axis.

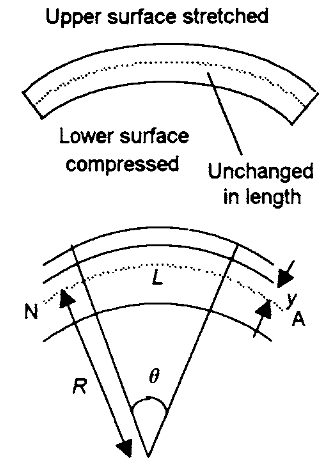


Figure 4.30 *Bending stretches the upper surface and contracts the lower surface, in-between there is an unchanged in length surface*

An initially horizontal plane through the beam which is a distance y from the neutral axis changes in length as a consequence of the beam being bent and the strain it experiences is the change in length ΔL divided by its initial unstrained length L. For circular arcs, the arc length is the radius of the arc multiplied by the angle it subtends, and thus, $L + \Delta L = (R + y)\theta$. The neutral axis NA will, by definition, be unstrained and so for it we have $L = R\theta$. Hence, the strain on aa is:

$$\text{strain} = \frac{\Delta L}{L} = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

Provided we can use Hooke's law, the stress due to bending which is acting this plane is:

$$\text{stress } \sigma = E \times \text{strain} = \frac{Ey}{R}$$

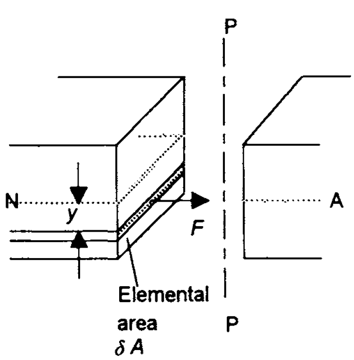


Figure 4.31 *Elemental area in a section PP*

Looking at a cross-sectional slice of the beam cut by PP we have Figure 4.31. The moment M of the elemental force F about the neutral axis is Fy and the stress σ acting on the elemental area is $F/\delta A$. Therefore the moment is $(\sigma\,\delta A)y$. Hence, using the equation we derived above for the stress, the moment of this element about the neutral axis is:

$$\text{moment} = \frac{Ey}{R}\delta A \times y = \frac{E}{R}y^2\delta A$$

The total moment M produced over the entire cross-section is the sum of all the moments produced by all the elements of area in the cross-section. Thus, if we consider each such element of area to be infinitesimally small, we can write:

$$M = \int \frac{E}{R}y^2\,dA = \frac{E}{R}\int y^2 dA$$

The integral is termed the *second moment of area* I of the section:

$$I = \int y^2\,dA$$

Thus we can write:

$$M = \frac{EI}{R}$$

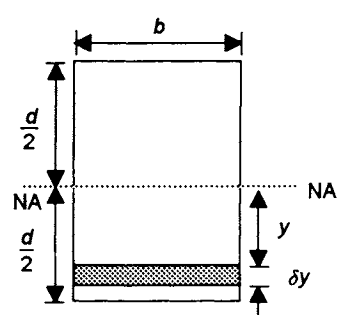


Figure 4.32 *Second moment of area*

For a rectangular cross-section of breadth b and depth d (Figure 4.32) with a segment of thickness δy a distance y from the neutral axis, the second moment of area for the segment is:

$$\text{second moment of area of strip} = y^2\delta A = y^2 b\delta y$$

The total second moment of area for the section about the neutral axis is thus:

$$\text{second moment of area} = \int_{-d/2}^{d/2} y^2 b\,dy = \frac{bd^3}{12}$$

4.2.3 Techniques for integration

There are a number of techniques which can aid in the integration of functions. In this section we look at integration by substitution, integration by parts and partial fractions.

Integration by substitution

This involves simplifying integrals by making a *substitution*. The term *integration by change of variable* is often used since the variable has to be changed as a result of the substitution. The aim of making a substitution is to put the integral into a simpler form for integration. As an illustration, consider the integral:

$$\int e^{5x}\,dx$$

Key points

Commonly used substitutions:

$\int f(ax+b)\,dx$ Let $u = ax + b$

$\int fx(ax^2+b)\,dx$ Let $u = ax^2 + b$

$\int \cos^m ax \sin^n ax\,dx$, when n is od
Let $u = \cos ax$. Use $\sin^2 ax + \cos^2 ax = 1$ in the simplification.

$\int \cos^m ax \sin^n ax\,dx$, when m is od
Let $u = \cos ax$. Use $\sin^2 ax + \cos^2 ax = 1$ in the simplification.

$\int \cos^m ax \sin^n ax\,dx$, when m and are both even or both odd
Rewrite the integral using:
$\sin^2 x = ½(1 - \cos 2x)$,
$\cos^2 x = ½(1 + \cos 2x)$,
$\sin x \cos x = ½ \sin 2x$.

The substitution $u = 5x$ reduces e^{5x} to e^u. However, we also need to change dx in the variable to du for the integration. Since du/dx = 5, we can write the integral as:

$$\int e^u \frac{du}{5} = \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

In the above case the substitution of u for $5x$ seemed a sensible way to simplify the integral. However, there are no general rules for finding suitable substitutions and the key points show some of the more commonly used substitutions.

Example

Determine the indefinite integral $\int (4x+1)^3\,dx$.

If we let $u = 4x + 1$ then du/dx = 4 and dx = ¼ du:

$$\int (4x+1)^3\,dx = \int u^3 \frac{1}{4}\,du = \frac{1}{4}\frac{u^4}{4} + C = \frac{1}{16}(4x+1)^4 + C$$

Example

Determine the indefinite integral $\int \frac{x}{3x^2+4}\,dx$.

If we let $u = 3x^2 + 4$, then du/dx = $6x$ and so x dx = (1/6) du. Hence:

$$\int \frac{x}{3x^2+4}\,dx = \int \frac{1}{u}\frac{1}{6}\,du = \frac{1}{6}\int \frac{1}{u}\,du$$

$$= \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln(3x^2+4) + C$$

The modulus sign is used with the integration of $1/u$ because no assumption is made at that stage as to whether u is positive or negative. The sign is dropped when the substitution is made because $3x^2 + 4$ is always positive.

Example

Determine the indefinite integral $\int \cos^2 x \sin^3\,dx$.

If we let $u = \cos x$, then du/dx = $-\sin x$ and so $\sin x$ dx = $-$du. The integral then can be written as:

$$\int \cos^2 x \sin^3 x \, dx = \int \cos^2 x \sin^2 x \sin x \, dx$$

$$= \int \cos^2 x(1 - \cos^2 x) \sin x \, dx$$

$$= \int u^2(1 - u^2) \, du = \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \tfrac{1}{3}\cos^3 x - \tfrac{1}{5}\cos^5 x + C$$

Example

Determine the indefinite integral $\int \cos x \sin^2 x \, dx$.

Let $u = \sin x$. Then $du/dx = \sin x$ and so $\sin x \, dx = du$. The integral can then be written as:

$$\int \cos x \sin^2 x \, dx = \int \sin^2 x \cos x \, dx$$

$$= \int u^2 \, du = \frac{u^3}{3} + C = \tfrac{1}{3}\sin^3 x + C$$

Trigonometric substitutions

A useful group of substitutions is to use trigonometric functions. For example, for integrals involving $\sqrt{(a^2 - x^2)}$ terms, we can use the substitution $x = a \sin \theta$. Then $\sqrt{(a^2 - x^2)} = \sqrt{(a^2 - a^2 \sin^2 \theta)} = a \cos \theta$, since we have $1 - \sin^2 \theta = \cos^2 \theta$. Since $dx/d\theta = a \cos \theta$ then $a \cos \theta \, d\theta$. The key points give other such substitutions.

Key points

Useful trigonometric substitutions:

$\int \sqrt{(a^2 - x^2)} \, dx$ Let $x = a \sin \theta$

$\int \sqrt{(a^2 + x^2)} \, dx$ Let $x = a \tan \theta$

$\int \sqrt{(x^2 - a^2)} \, dx$ Let $x = a \sec \theta$

Example

Determine the indefinite integral $\int \sqrt{(1 - x^2)} \, dx$.

Let $x = \sin \theta$. Then $\sqrt{(1 - x^2)} = \sqrt{(1 - \sin^2 \theta)} = \cos \theta$. Since $dx/d\theta = \cos \theta$ then $dx = \cos \theta \, d\theta$. Thus the integral becomes:

$$\int \sqrt{(1 - x^2)} \, dx = \int \cos\theta \cos\theta \, d\theta = \int \cos^2\theta \, d\theta$$

Since $\cos 2\theta = 2\cos^2 \theta - 1$, we have:

$$\int \tfrac{1}{2}(1 + \cos 2\theta) \, d\theta = \tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C$$

Back substitution using $\theta = \sin^{-1} x$ gives:

$$\tfrac{1}{2}\sin^{-1} x + \tfrac{1}{4}\sin(2\sin^{-1} x) + C$$

However, a simpler expression is obtained if we first replace the sin 2θ using sin 2θ = 2 sin θ cos θ = 2sinθ √(1 − sin² θ).

$$\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta + C = \tfrac{1}{2}\sin^{-1}x + \tfrac{1}{2}x\sqrt{1-x^2} + C$$

Example

Determine the indefinite integral $\int \frac{1}{x^2+4}\,dx$.

Let $x = 2\tan\theta$. Then $dx/d\theta = 2\sec^2\theta$ and so:

$$\int \frac{1}{x^2+4}\,dx = \int \frac{1}{4\tan^2\theta+4} 2\sec^2\theta\,d\theta$$

$$= \int \frac{1}{4\sec^2\theta} 2\sec^2\theta\,d\theta$$

$$= \int \tfrac{1}{2}\,d\theta = \tfrac{1}{2}\theta + C = \tfrac{1}{2}\tan^{-1}\frac{x}{2} + C$$

Another form of useful substitution, when we have integrals involving sin x, cos x, tan x terms, is to let $u = \tan \tfrac{1}{2}x$. Then $du/dx = \tfrac{1}{2}\sec^2 \tfrac{1}{2}x$. But $\sec^2 x = 1 + \tan^2 x$, thus $du/dx = \tfrac{1}{2}(1 + \tan^2 x) = \tfrac{1}{2}(1 + u^2)$. Thus $dx = 2\,du/(1 + u^2)$. The trigonometric functions can all be expressed in terms of u. Thus:

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\sin\frac{x}{2}\cos\frac{x}{2}\frac{\cos\frac{x}{2}}{\cos\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\sec^2\frac{x}{2}}$$

$$= \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2u}{1+u^2}$$

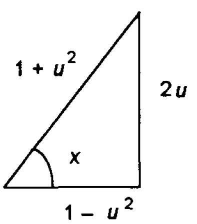


Figure 4.33 *Angle x*

Figure 4.33 shows the right-angled triangle with such an angle. Hence:

$$\cos x = \frac{1-u^2}{1+u^2} \quad \text{and} \quad \tan x = \frac{2u}{1-u^2}$$

Note that integration of the squares of trigonometric functions can be obtained by using trigonometric identities to put the functions in non-squared form. Thus:

$$\int \sin^2 x\,dx = \int \tfrac{1}{2}(1-\cos 2x)\,dx = \tfrac{1}{2}(x - \tfrac{1}{2}\sin 2x) + C$$

$$\int \cos^2 x\,dx = \int \tfrac{1}{2}(1+\cos 2x)\,dx = \tfrac{1}{2}(x + \tfrac{1}{2}\sin 2x) + C$$

$$\int \tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x + C$$

Example

Determine the indefinite integral $\int \frac{1}{\sin x}\,dx$.

Let $u = \tan \frac{1}{2}x$, then $du/dx = \frac{1}{2}\sec^2 \frac{1}{2}x = \frac{1}{2}(1 + \tan^2 \frac{1}{2}x) = \frac{1}{2}(1 + u^2)$ and replacing $\sin x$ by $2u/(1 + u^2)$:

$$\int \frac{1}{\sin x}\,dx = \int \frac{1+u^2}{2u}\frac{2}{1+u^2}\,du = \int \frac{1}{u}\,du$$

$$= \ln|u| + C = \ln\left|\tan\frac{x}{2}\right| + C$$

Substitution with definite integrals

The above has discussed the substitution procedure with indefinite integrals where the variable was changed from x to u. When we have definite integrals we can do the same procedure and take account of the limits of integration at the end *after* reversing the substitution. The limits are in terms of values of x. However, it is often simpler to express the limits in terms of u and take account of the limits *before* reversing the substitution. To illustrate this, consider the integration of $\cos^3 x$ between the limits 0 and $\frac{1}{2}\pi$. If we let $u = \sin x$ then $du/dx = \cos x$ and so $\cos x\,dx = du$. When $x = 0$ then $u = 0$ and when $x = \frac{1}{2}\pi$ then $u = 1$. Thus the integral can be written as:

$$\int_0^{\pi/2} \cos^3 x\,dx = \int_0^{\pi/2} \cos^2 x \cos x\,dx = \int_0^{\pi/2} (1 - \sin^2 x)\cos x\,dx$$

$$= \int_0^1 (1 - u^2)\,du = \left[u - \frac{u^3}{3}\right]_0^1 = \frac{2}{3}$$

Integration by parts

The product rule for differentiation gives:

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$

Integrating both sides of this equation with respect to x gives:

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f(x)\frac{d}{dx}g(x)\,dx + \int g(x)\frac{d}{dx}f(x)\,dx$$

Hence:

$$\int f(x)\frac{d}{dx}g(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int g(x)\frac{d}{dx}f(x)\,dx$$

$$= f(x)g(x) - \int g(x)\frac{d}{dx}f(x)\,dx \qquad [31]$$

This is the formula for *integration by parts.* This is often written in terms of $u = f(x)$ and $v = g(x)$ as:

Key point

Integration by parts:

$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \qquad [32]$$

With a definite integral the equation becomes:

$$\int_a^b u \frac{dv}{dx} dx = [uv]_a^b - \int_a^b v \frac{du}{dx} dx \qquad [33]$$

Example

Determine the indefinite integral $\int x\, e^x\, dx$.

The integral consists of the product of two factors. If we let $u = x$ and $dv/dx = e^x$, then $v = \int e^x\, dx$ and equation [32] gives:

$$\int x\, e^x\, dx = x\, e^x - \int (e^x)(1)\, dx = x\, e^x - e^x + C$$

Example

Determine the indefinite integral $\int e^x \sin x\, dx$.

Let $u = e^x$ and $dv/dx = \sin x$. Then:

$$v = \int \sin x\, dx = -\cos$$

Hence, using equation [32] gives:

$$\int e^x \sin x\, dx = e^x(-\cos x) - \int (-\cos x)(e^x)\, dx$$

$$= -e^x \cos x + \int e^x \cos x\, dx + C$$

Applying integration by parts again, with $u = e^x$ and $dv/dx = \cos x$. Then $v = \int \cos x\, dx = \sin x$. Hence, using equation [32] gives:

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx + C$$

Thus:

$$\int e^x \sin x\, dx = \tfrac{1}{2}(-e^x \cos x + e^x \sin x + C)$$

Example

Determine the definite integral $\int_0^1 x^2\, e^x\, dx$.

Let $u = x^2$ and $dv/dx = e^x$. Then $v = \int e^x\,dx = e^x$. Thus, using equation [9]:

$$\int_0^1 x^2 e^x\,dx = [x^2 e^x]_0^1 - \int_0^1 e^x(2x)\,dx$$

Applying integration by parts again, with $u = x$ and $dv/dx = e^x$. Then $v = \int e^x\,dx = e^x$. Thus, using equation [33]:

$$\int_0^1 x^2 e^x\,dx = [x^2 e^x]_0^1 - 2[x e^x]_0^1 + 2\int_0^1 e^x\,dx$$

$$= [x^2 e^x]_0^1 - 2[x e^x]_0^1 + 2[e^x]_0^1$$

$$= e^1 - 2\,e^1 + 2\,e^1 - 2\,e^0 = e - 2$$

Integration by partial fractions

Integrals involving fractions can often by simplified by expressing the integral as the sum or difference of two or more partial fractions which then lend themselves to easier integration. For example:

$$\frac{3x+4}{x^2+3x+2} = \frac{3x+4}{(x+1)(x+2)}$$

can be expressed as the partial fractions:

$$\frac{3x+4}{(x+1)(x+2)} = \frac{1}{x+1} + \frac{1}{x+2}$$

When the degree of the denominator is greater than that of the numerator then an expression can be directly resolved into partial fractions. The form taken by the partial fractions depends on the type of denominator concerned.

- If the denominator contains a *linear factor*, i.e. a factor of the form $(x + a)$, then for each such factor there will be a partial fraction of the form:

  $$\frac{A}{(x+a)}$$

  where A is some constant.

- If the denominator contains *repeated linear factors*, i.e. a factor of the form $(x + a)^n$, then there will be partial fractions:

  $$\frac{A}{(x+a)} + \frac{B}{(x+a)^2} + \ldots + \frac{C}{(x+a)^n}$$

  with one partial fraction for each power of $(x + a)$.

Key points

The procedure for obtaining partial fractions can be summarised as:

1. If the degree of the denominator is equal to, or less than, that of the numerator, divide the denominator into the numerator to obtain the sum of a polynomial plus a fraction which has the degree of the denominator greater than that of the numerator.

2. Write the denominator in the form of linear factors, i.e. of the form $(ax + b)$, or irreducible quadratic factors, i.e. of the form $(ax^2 + bx + c)$.

3. Write the fraction as a sum of partial fractions involving constants A, B, etc.

4. Determine the unknown constants which occur with the partial fractions by equating the fraction with the partial fractions and either solving the equation for specific values of x or equating the coefficients of equal powers of x.

5. Replace the constants in the partial fractions with their values.

- If the denominator contains an *irreducible quadratic factor*, i.e. a factor of the form $ax^2 + bx + c$, then there will be a partial fraction of the form:

  $$\frac{Ax+B}{ax^2+bx+c}$$

  for each such factor.

- If the denominator contains *repeated quadratic factors*, i.e. a factor of the form $(ax^2 + bx + c)^n$, there will be partial fractions of the form:

  $$\frac{Ax+B}{ax^2+bx+c}+\frac{Cx+D}{(ax^2+bx+c)^2}+\ldots+\frac{Ex+F}{(ax^2+bx+c)^n}$$

  with one for each power of the quadratic.

The values of the constants A, B, C, etc. can be found by either making use of the fact that the equality between the fraction and its partial fractions must be true for all values of the variable x or that the coefficients of x^n in the fraction must equal those of x^n when the partial fractions are multiplied out.

When the degree of the denominator, i.e. the power of its highest term, is equal to or less than that of the numerator, the denominator must be divided into the numerator until the result is the sum of terms with the remainder fraction term having a denominator which is of higher degree than its numerator. Consider, for example, the fraction:

$$\frac{x^3-x^2-3x+1}{x^2-3x+2}$$

The numerator has a degree of 3 and the denominator a degree of 2. Thus, dividing has to be used. Thus

$$\frac{x^3-x^2-3x+1}{x^2-3x+2}=x+2+\frac{x-3}{x^2-3x+2}$$

The fractional term can then be simplified using partial fractions.

$$\frac{x-3}{x^2-3x+2}=\frac{x-3}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

to give:

$$\frac{x^3-x^2-3x+1}{x^2-3x+2}=x+2+\frac{2}{x-1}-\frac{1}{x-2}$$

Example

Simplify into its partial fraction form: $\frac{3x+4}{(x+1)(x+2)}$.

This has two linear factors in the denominator and so the partial fractions are of the form:

$$\frac{A}{x+1}+\frac{B}{x+2}$$

with one partial fraction for each linear term. Thus for the expressions to be equal we must have:

$$\frac{3x+4}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Thus

$3x + 4 = A(x + 2) + B(x + 1)$

Consider the requirement that this relationship is true for all values of x. Then, when $x = -1$ we must have:

$-3 + 4 = A(-1 + 2) + B(-1 + 1)$

Hence $A = 1$. When $x = -2$ we must have:

$-6 + 4 = A(-2 + 2) + B(-2 + 1)$

Hence $B = 2$.

Alternatively, we could have determined these constants by multiplying out the expression and considering the coefficients, i.e.

$3x + 4 = A(x + 2) + B(x + 1) = Ax + 2A + Bx + B$

Thus, for the coefficients of x to be equal we must have $3 = A + B$ and for the constants to be equal $4 = 2A + B$. These two simultaneous equations can be solved to give A and B. The partial fractions are thus:

$$\frac{3x+4}{(x+1)(x+2)}=\frac{1}{x+1}+\frac{2}{x+2}$$

Example

Determine the indefinite integral $\int \frac{1}{x^2-1}\,dx$.

The fraction $1/(x^2 - 1)$ can be written as:

$$\frac{1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}=\frac{A(x+1)+B(x-1)}{(x-1)(x+1)}$$

Hence, equating coefficients of x gives $A + B = 0$ and equating integers gives $A - B = 1$. Thus $A = ½$ and $B = -½$. Hence the integral can be expressed as:

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$$

We can determine these integrals by substitution. Thus if we let $u = x - 1$ then $du/dx = 1$ and so:

$$\int \frac{1}{x-1}\,dx = \int \frac{1}{u}\,du = \ln|u| = \ln|x-1| + A$$

Likewise the integral of $1/(x + 1)$ is $\ln|x + 1| + B$. Hence:

$$\int \frac{1}{x^2-1}\,dx = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C$$

Example

Determine the indefinite integral $\int \frac{x^3}{x-2}\,dx$.

This fraction has a numerator of higher degree than the denominator and so the numerator must be divided by the denominator until the remainder is of lower degree than the denominator. Thus:

$$\begin{array}{r|l}
 & x^2 + 2x + 4 \\
\hline
x-2 & x^3 \\
 & \underline{x^3 - 2x^2} \\
 & \quad 2x^2 \\
 & \quad \underline{2x^2 - 4x} \\
 & \qquad\quad 4x \\
 & \qquad\quad \underline{4x - 8} \\
 & \qquad\qquad\ 8
\end{array}$$

Hence the integral becomes:

$$\int \frac{x^3}{x-2}\,dx = \int\left(x^2 + 2x + 4 + \frac{8}{x-2}\right)dx$$

$$= \frac{x^3}{3} + x^2 + 4x + 8\ln|x-2| + C$$

Example

Determine the indefinite integral $\int \frac{1}{x(x^2+1)}\,dx$.

Expressed as partial fractions:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)}$$

Equating the constant terms gives $A = 1$. Equating the coefficients of x gives $C = 0$. Equating the coefficients of x^2 gives $A + B = 0$, and so $B = -1$. Thus the integral becomes:

$$\int \frac{1}{x(x^2+1)}\,dx = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right) dx$$

The integration of $1/(x^2 + 1)$ can be carried out by using a substitution. Let $u = x^2 + 1$ and so $du/dx = 2x$. Thus:

$$\int \frac{x}{x^2+1}\,dx = \int \frac{1}{2u}\,du = \tfrac{1}{2}\ln|u| + C = \tfrac{1}{2}\ln|x^2+1| +$$

and so:

$$\int \frac{1}{x(x^2+1)}\,dx = \ln|x| - \tfrac{1}{2}\ln|x^2+1| + C$$

Maths in action

The technique of using partial fractions to simplify expressions has many uses. In Chapter 6, we shall see how partial fractions can help in the solution of differential equations using the Laplace transform. As an illustration, consider a differential equation relating rotational displacement θ to time t for a rotating power transmission shaft:

$$\frac{d^2\theta}{dt^2} - 6\frac{d\theta}{dt} - 10\theta = 20 - e^{2t}$$

Given that when $t = 0$ we have $\theta = 4$ and $d\theta/dt = 25/2$, the Laplace transform enables the differential equation to be written in the form:

$$\mathcal{L}\{\theta\} = \frac{4s^3 - \frac{39}{2}s^2 + 42s - 40}{s(s-2)(s^2-6s+10)}$$

We can use the method of partial fractions to simplify the expression. Let the fraction be replaced by:

$$\frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2-6s+10}$$

Then we must have:

$$4s^3 - \tfrac{39}{2}s^2 + 42s - 40 = A(s-2)(s^2-6s+10) + B(s)(s^2-6s+10) + (Cs+D)(s)(s-2)$$

If we let $s = 2$, then $32 - 78 + 84 - 40 = B(2)(4 - 12 + 10)$ and so $B = -1/2$. If we let $s = 0$, then $-40 = A(-2)(10)$ and so $A = 2$. Comparing coefficients of s gives $42 = 22A + 10B - 2D$ and so $D = -3/2$. Comparing coefficients of s^3 gives $4 = A + B + C$ and so $C = 5/2$. Putting these values into the partial fraction equation gives:

$$\frac{2}{s} + \frac{\left(-\frac{1}{2}\right)}{s-2} + \frac{\left(\frac{5}{2}\right)s + \left(-\frac{3}{2}\right)}{s^2-6s+10}$$

$$= \frac{2}{s} - \frac{1}{2(s-2)} + \frac{5s-3}{2(s^2-6s+10)}$$

This is a lot easier to handle than the original equation.

4.2.4 Means

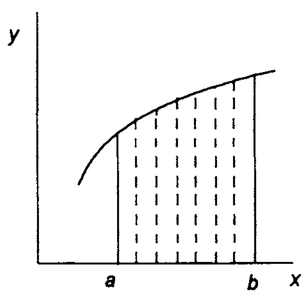


Figure 4.34 *Mean value*

The *mean* of a set of numbers is their sum divided by the number of numbers summed. The *mean value of a function* between $x = a$ and $x = b$ is the mean value of all the ordinates between these limits. Suppose we divide the area into n equal width strips (Figure 4.34), then if the values of the mid-ordinates of the strips are $y_1, y_2, \ldots y_n$ the mean value is:

$$\text{mean value of } y = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

If δx is the width of the strips, then $n\,\delta x = b - a$. Thus:

$$\text{mean value of } y = \frac{\left(y_1 + y_2 + \ldots + y_n\right)\delta x}{b-a}$$

Hence, as $\delta x \to 0$:

$$\text{mean value of } y = \frac{1}{b-a}\int_a^b y\,dx \qquad [34]$$

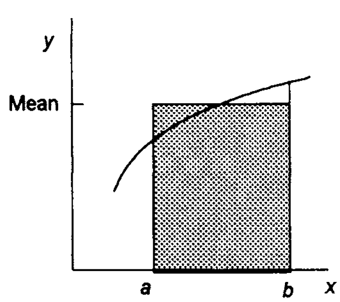


Figure 4.35 *Mean value rectangle*

Since the sum of all the $y\,\delta x$ terms is the area under the graph between $x = a$ and $x = b$:

$$\text{mean value of } y = \frac{\text{area under graph}}{b-a}$$

But the product of the mean value and $(b - a)$ is the area of a rectangle of height equal to the mean value and width $(b - a)$. Figure 4.35 shows this mean value rectangle.

Example

Determine the mean value of the function $y = \sin x$ between $x = 0$ and $x = \pi$.

The mean value of function is:

$$\frac{1}{b-a}\int_a^b y\,dx = \frac{1}{\pi-0}\int_0^\pi \sin x\,dx$$

$$= \frac{1}{\pi}[-\cos x]_0^\pi = \frac{2}{\pi} = 0.637$$

Root-mean-square values

The power dissipated by an alternating current i when passing through a resistance R is i^2R. The mean power dissipated over a time interval from $t = 0$ to $t = T$ will thus be:

$$\text{mean power} = \frac{1}{T-0}\int_0^T i^2R\,dt = \frac{R}{T}\int_0^T i^2\,dt$$

If we had a direct current I generating the same power then we would have:

$$I^2R = \frac{R}{T}\int_0^T i^2\,dt$$

and:

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} \qquad [35]$$

This current I is known as the *root-mean-square* current. There are other situations in engineering and science where we are concerned with determining root-mean-square quantities. The procedure is thus to determine the mean value of the squared function over the required interval and then take the square root.

Example

Determine the root-mean-square current value of the alternating current $i = I \sin \omega t$ over the time interval $t = 0$ to $t = 2\pi/\omega$.

The root-mean-square value is:

$$I = \sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{\omega}{2\pi}\int_0^{2\pi/\omega} I^2\sin^2\omega t\,dt}$$

$$= \sqrt{\frac{I^2\omega}{2\pi}\int_0^{2\pi/\omega} \tfrac{1}{2}(1-\cos 2\omega t)\,dt}$$

$$= \sqrt{\frac{I^2\omega}{4\pi}\left[t - \frac{1}{2\omega}\sin 2\omega t\right]_0^{2\pi/\omega}} = \frac{I}{\sqrt{2}}$$

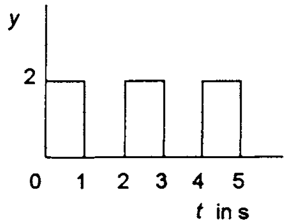


Figure 4.36 *Example*

Example

Determine the root-mean-square value of the waveform shown in Figure 4.36 over a period of 0 to 2 s.

From $t = 0$ to $t = 1$ s the waveform is described by $y = 2$. From $t = 1$ s to $t = 2$ s the waveform is described by $y = 0$. Thus the root-mean-square value is given by:

$$y_{rms} = \sqrt{\tfrac{1}{2}\left(\int_0^1 4\,dt + \int_1^2 0\,dt\right)} = \sqrt{\tfrac{1}{2}[4t]_0^1} = \sqrt{2}$$

Example

An alternating current is defined by the equation:

$i = 25 \sin 100\pi t$ mA

Determine its mean value over half-a-cycle and the root-mean-square value over a cycle.

We have $100\pi = 2\pi f = 2\pi/T$, where f is the frequency and T the periodic time. Hence the periodic time is 0.02 s and the time for half-a-cycle is 0.01 s.

Using equation [34], the mean value over half-a-cycle is:

$$\frac{1}{0.01 - 0}\int_{t=0}^{t=0.01} 25 \sin 100\pi t\,dt$$

We can use the standard form for the integral of sin ax to give the mean value as:

$$\frac{25}{0.01}\left[-\frac{1}{100\pi}\cos 100\pi t + C\right]_0^{0.01}$$

$$= \frac{1}{0.04\pi}\Big((-\cos\pi) - (-\cos 0)\Big) = \frac{1}{0.04\pi}(1 + 1)$$

Thus the mean value over half-a-cycle is 15.92 mA. Note that for a sinusoidal signal the mean value over a full cycle is zero.

The root-mean-square value over a cycle is given by equation [35] as:

$$\sqrt{\frac{1}{T}\int_0^T i^2\,dt} = \sqrt{\frac{1}{0.02}\int_0^{0.02} 25^2 \sin^2 100\pi t\,dt}$$

Since $\sin^2\theta = ½(1 - \cos 2\theta)$, we can write:

$$\sqrt{\frac{625}{0.02}\int_0^{0.02} \tfrac{1}{2}(1 - \cos 200\pi t\,dt}$$

$$\sqrt{\frac{625}{0.04}\left[t - \frac{1}{200\pi}\sin 200\pi t\right]_0^{0.02}} = \sqrt{\frac{625}{0.04}\times 0.02}$$

and so the r.m.s. current $25/\sqrt{2}$ = 17.68 mA. Note, that in general, the root-mean-square value of a sinusoidal signal is always the maximum value divided by $\sqrt{2}$.

Problems 4.2

1 Determine the integrals of the following:

(a) 4, (b) $2x^3$, (c) $2x^3 + 5x$, (d) $x^{2/3} - 3x^{1/2}$, (e) $4 + \cos 5x$,

(f) $2\,e^{-3x}$, (g) $4\,e^{x/2} + x^2 + 2$, (h) $4/x$

2 Determine the areas under the following curves between the specified limits and the x-axis:

(a) $y = 4x^3$ between $x = 1$ and $x = 2$,
(b) $y = x$ between $x = 0$ and $x = 4$,
(c) $y = 1/x$ between $x = 1$ and $x = 3$,
(d) $y = x^3 - 3x^2 - 2x + 2$ between $x = -1$ and $x = 2$,
(e) $y = x^2 - x - 2$ between $x = -1$ and $x = 2$,
(f) $y = x^2 - 1$ between $x = -1$ and $x = 2$,
(g) the area between $x = 0$ and $x = 2$ for the curve defined by $y = x^2$ between $x = 0$ and $x = 1$ and by $y = 2 - x$ between $x = 1$ and $x = 2$.

3 Determine the areas bounded by graphs of the following functions and between the specified ordinates:

(a) $y = 9 - x^2$, $y = -2$, $x = -2$ and $x = 2$,
(b) $y = 4$, $y = x^2$, $x = 0$ and $x = 1$

4 Determine the geometrical area enclosed between the graph of the function $y = x(x - 1)(x - 2)$ and the x-axis.
5 Determine the area bounded by graphs of $y = x^3$ and $y = x^2$.
6 Determine the area bounded by the graph of $y = \sin x$, the x-axis and the line $x = \pi/2$.
7 Determine the area bounded by graphs of $y = x^2 - 2x + 2$ and $y = 4 - x$.

8 Determine the values, if they exist, of the following definite integrals:

(a) $\int_{-\infty}^{1} x\,dx$, (b) $\int_{1}^{\infty} \frac{1}{x^3}\,dx$, (c) $\int_{0}^{\infty} e^{-3x}\,dx$, (d) $\int_{-\infty}^{-1} x^4\,dx$

9 Determine the following indefinite integral by using the given substitutions:

(a) $\int (x^2+1)x^3\,dx$ using $u = x^2+1$,

(b) $\int 2\,e^{4x-1}\,dx$ using $u = 4x-1$,

(c) $\int \frac{x+1}{\sqrt{2x+1}}\,dx$ using $u = \sqrt{2x+1}$,

(d) $\int x\sin x^2\,dx$ using $u = x^2$,

(e) $\int \sqrt{x^2+4}\,dx$ using $x = 2\sinh u$,

(f) $\int x\sqrt{x-1}\,dx$ using $u = \sqrt{x-1}$,

(g) $\int \sec x\,dx$ using $u = \tan\frac{1}{2}x$,

(h) $\int \frac{1}{\sqrt{9-x^2}}\,dx$ using $x = 3\sin\theta$,

(i) $\int \sin^2 2x\cos^3 2x\,dx$ using $u = \sin x$

10 Determine the following indefinite integrals by making appropriate substitutions:

(a) $\int x\sqrt{x+2}\,dx$, (b) $\int \frac{1}{(x^2+1)^{3/2}}\,dx$, (c) $\int \sin^3 x\,dx$,

(d) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$, (e) $\int x\sqrt{x^2+2}\,dx$, (f) $\int \frac{1}{4+25x^2}\,dx$,

(g) $\int \cos^3 x\sin^4 x\,dx$, (h) $\int \tan^3 x\sec^2 x\,dx$,

(i) $\int \frac{1}{5+4\cos x}\,dx$

11 By making appropriate substitutions, evaluate the following definite integrals:

(a) $\int_0^1 \frac{1}{2-x}\,dx$, (b) $\int_0^1 \frac{3x^2}{(x^3+9)^2}\,dx$, (c) $\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$,

(d) $\int_{-1}^{1} x^2\sqrt{2-x^2}\,dx$, (e) $\int_0^2 \frac{1}{4+x^2}\,dx$

12 Using the method of integration by parts, determine the following indefinite integrals:

(a) $\int x^2 \ln x \, dx$, (b) $\int x\, e^{2x} \, dx$, (c) $\int x^3 \cos x \, dx$,

(d) $\int x \sin 5x \, dx$ (e) $\int x \ln 3x \, dx$, (f) $\int \sin^2 x \, dx$

13 Using the method of integration by parts, evaluate the following definite integrals:

(a) $\int_0^{\pi/2} x \cos x \, dx$, (b) $\int_0^{\pi/2} x \cos^2 x \, dx$,

(c) $\int_0^{\pi} (\pi - x) \cos x \, dx$

14 Determine the following indefinite integrals:

(a) $\int \frac{x^2}{2x-3} \, dx$, (b) $\int \frac{x}{1-2x} \, dx$, (c) $\int \frac{x^2}{2x^2+x-3} \, dx$,

(d) $\int \frac{x^2}{(x^2-1)(2x+1)} \, dx$, (e) $\int \frac{x+1}{x(x-2)(x+2)} \, dx$,

(f) $\int \frac{3x-1}{(2x+1)(x-1)} \, dx$, (g) $\int \frac{2x^3+3x^2-3}{2x^2-x-1} \, d$

(h) $\int \frac{1}{(x-2)(x-3)} \, dx$, (i) $\int \frac{5x^2+20x+6}{x(x+1)^2} \, dx$,

(j) $\int \frac{2x^3-4x-8}{x(x-1)(x^2+4)} \, dx$, (k) $\int \frac{1}{x^2(x^2+1)} \, dx$

15 Determine the moment of inertia for a uniform triangular sheet of mass M, base b and height h about (a) an axis through the centroid and parallel to the base and (b) about the base. The centroid is at one-third the height.

16 Determine the moment of inertia of a flat circular ring with an inner radius r, outer radius $2r$ and mass M about an axis through its centre and at right angles to its plane.

17 Determine the moment of inertia of a uniform square sheet of mass M and side L about (a) an axis through its centre and in its plane, (b) an axis in its plane a distance d from its centre.

18 Determine the mean values of the following functions between the specified limits:

(a) $y = 2x$ between $x = 0$ and $x = 1$,
(b) $y = x^2$ between $x = 1$ and $x = 4$,
(c) $y = 3x^2 - 2x$ between $x = 1$ and $x = 4$,
(d) $y = \cos^2 x$ between $x = 0$ and $x = 2\pi$.

19 With simple harmonic motion, the displacement x of an object is related to the time t by $x = A \cos \omega t$. Determine the mean value of the displacement during one-quarter of an oscillation, i.e. between when $\omega t = 0$ and $\omega t = $ o.

20 The number N of radioactive atoms in a sample is a function of time t, being given by $N = N_0\, e^{-\lambda t}$. Determine

the mean number of radioactive atoms in the sample between $t = 0$ and $t = 1/\lambda$.

21 Determine the root-mean-square values of the following functions between the specified limits:

(a) $y = x^2$ from $x = 1$ to $x = 3$,
(b) $y = x$ from $x = 0$ to $x = 2$,
(c) $y = \sin x + 1$ from $x = 0$ to $x = 2\pi$,
(d) $y = \sin 2x$ from $x = 0$ to $x = \pi$,
(e) $y = e^x$ from $x = -1$ to $x = +1$

22 Determine the root-mean-square value of a half-wave rectified sinusoidal voltage. Between the times $t = 0$ and $t = \pi/\omega$ the equation is $v = V \sin \omega t$ and between $t = \pi/\omega$ and $t = 2\pi/\omega$ we have $v = 0$.

5 Differential equations

Summary

This chapter introduces ordinary differential equations, shows how they can be used to model the behaviour of systems in engineering and looks at their solution for different inputs to the systems. Differential equations arise from such situations as the lumped models designed to represent systems (see Chapter 3), the motion of projectiles, the cooling of a solid or liquid, transient currents and voltages in electrical circuits, oscillations with mechanical or electrical systems and the rate of decay of radioactive substances.

Objectives

By the end of this chapter, the reader should be able to:

- represent engineering systems by differential equations;
- solve first- and second-order differential equations;
- solve the differential equations representing models of engineering systems for step and ramp inputs.

5.1 Differential equations

A *differential equation* is an equation involving derivatives of a function. Thus examples of differential equations are:

$$\frac{dy}{dx} + 2y = 5 \text{ and } \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 5$$

The term *ordinary differential equation* is used when there is only one independent variable, the above examples having only y as a function of x and so being ordinary differential equations.

Chapter 3 showed how differential equations can be evolved for the mathematical models of lumped engineering systems. The following extends that analysis to illustrate how ordinary differential equations can be evolved for some simple systems.

Mechanical systems

Consider a freely falling body of mass m in air (Figure 5.1). The gravitational force acting on the body is mg, where g is the

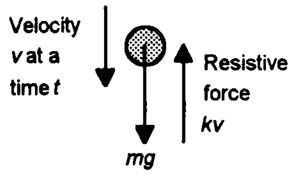


Figure 5.1 *Body falling in air*

Key point

The order of a differential equation is equal to the order of the highest derivative that appears in the equation.

acceleration due to gravity. Opposing the movement of the body through the air is air resistance. Assuming that the air resistance force is proportional to the velocity v, the net force F acting on the body is $mg - kv$, where k is a constant. But Newton's second law gives the net force F acting on a body as the product of its mass m and acceleration a, i.e. $F = ma$. But acceleration is the rate of change of velocity v with time t. Thus we can write:

$$F = m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv$$

and so the differential equation describing this system is:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} + kv = mg \qquad [1]$$

This is a *first-order differential equation* because the highest derivative is $\mathrm{d}v/\mathrm{d}t$. It describes how the velocity varies with time.

Consider another mechanical system, an object of mass m suspended from a support by a spring (Figure 5.2). When the mass is placed on the spring it stretches by d, called the static displacement (Figure 5.2(a)). Assuming Hooke's law, and so the displacement proportional to the force exerted by the spring, we can write $F = kd$. At equilibrium, considering the vertical forces and applying $F = ma$, we have $mg - kd = ma = 0$ as there is zero acceleration. Now if we pull the body down a distance x from this equilibrium position (Figure 5.2(b)) and again apply $F = ma$ to the system when released, the net restoring force acting on the body is $mg - k(d + x) = ma$ and so $-kx = ma$. Since acceleration is the rate of change of velocity with time, with velocity being the rate of change of displacement with time:

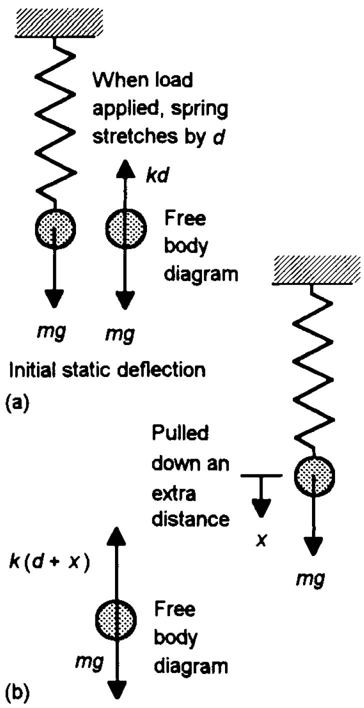


Figure 5.2 *Mass on spring*

$$ma = m\frac{\mathrm{d}v}{\mathrm{d}t} = m\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

and so:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + kx = 0 \qquad [2]$$

This is a *second-order differential equation* because the highest derivative is $\mathrm{d}^2x/\mathrm{d}t^2$. It describes the resulting oscillations of the body after it has been released.

Electrical systems

For an electrical circuit with a resistor in series with a capacitor (Figure 5.3), the supply voltage V equals the sum of the voltages across the resistor and capacitor:

$$V = v_R + v_C = Ri + v_C$$

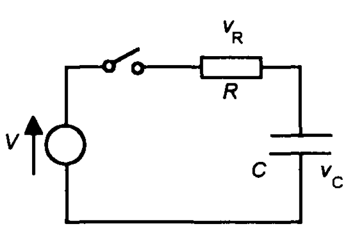


Figure 5.3 *Series RC circuit*

When a pure capacitor has a potential difference v applied across it, the charge q on the plates is given by $q = Cv$, where C is the capacitance. Current i is the rate of movement of charge and so:

$$i = \frac{dq}{dt} = C\frac{dv}{dt} \quad [3]$$

and thus we have:

$$RC\frac{dv_C}{dt} + v_C = V \quad [4]$$

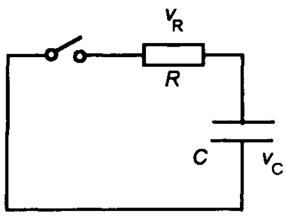


Figure 5.4 *RC discharge circuit*

This first-order differential equation describes how the capacitor voltage changes with time from when the switch is closed.

When a charged capacitor discharges through a resistance (Figure 5.4) then $v_R + v_C = 0$ and so:

$$RC\frac{dv_C}{dt} + v_C = 0 \quad [5]$$

This first-order differential equation describes how the capacitor voltage changes with time from when the switch is closed.

When a pure inductor has a current i flowing through it, then the induced e.m.f. produced in the component is proportional to the rate of change of current, the induced e.m.f. being $-L\, di/dt$ where L is the inductance. If the component has only inductance and no resistance, then there can be no potential drop across the component due to the current through the resistance and thus to maintain the current through the inductor the voltage source must supply a potential difference v which just cancels out the induced e.m.f. Thus the potential difference across an inductor is:

$$v = L\frac{di}{dt} \quad [6]$$

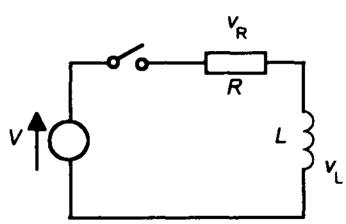


Figure 5.5 *Series RL circuit*

If we have an electrical circuit containing an inductor in series with a resistor (Figure 5.5) then, when the supply voltage V is applied, we have $V = v_L + v_R$. Thus, using equation [6]:

$$L\frac{di}{dt} + Ri = V$$

The steady state current I will be attained when the current ceases to change with time. We then have $RI = V$ and so the first-order differential equation can be written as:

$$\frac{L}{R}\frac{di}{dt} + i = I \quad [7]$$

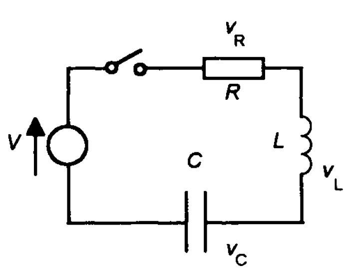


Figure 5.6 *Series RLC circuit*

Consider now a circuit including a resistor, a capacitor and an inductor in series (Figure 5.6). When the switch is closed the supply voltage v is applied across the three components and $V = v_R + v_L + v_C$. Thus, using equation [6]:

$$Ri + L\frac{di}{dt} + v_C = V$$

Since $i = C\, dv_C/dt$ (equation [3]), then:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V \qquad [8]$$

This second-order differential equation describes how the voltage across the capacitor varies with time.

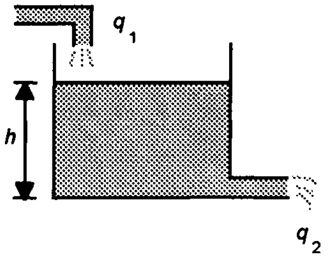


Figure 5.7 *Liquid level in a tank*

Hydraulic systems

Consider an open tank into which liquid can enter at the top through one pipe and leave at the base through another (Figure 5.7). If the liquid enters at the rate of a volume of q_1 per second and leaves at the rate of q_2 per second, then the rate at which the volume V of liquid in the tank changes with time is:

$$\frac{dV}{dt} = q_1 - q_2$$

But $V = Ah$, where A is the cross-sectional area of the tank and h the height of the liquid in the tank. Thus:

$$\frac{d(Ah)}{dt} = A\frac{dh}{dt} = q_1 - q_2$$

The rate at which liquid leaves the tank, when flowing from the base of the tank into the atmosphere, is given by Torricelli's theorem as $q_2 = \sqrt{(2gh)}$. Thus we have:

$$A\frac{dh}{dt} + \sqrt{2gh} = q_1 \qquad [9]$$

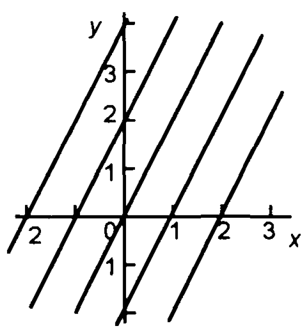


Figure 5.8 *General solution*

Solving differential equations

The differential equation $dy/dx = 2$ describes a straight line with a constant gradient of 2 (Figure 5.8). There are, however, many possible graphs which fit this specification, the family of such lines having equations of the form $y = 2x + A$, where A is a constant. These are all solutions for the differential equation.

Thus the differential equation $dy/dx = 2$ has many solutions given by $y = 2x + A$, this being termed the *general solution*. Only if constraints are specified which enable constants like A to be evaluated will there be just one solution, this being then termed a *particular solution*. The term *initial conditions* are used for the constraints if specified at $y = 0$ and *boundary conditions* if specified at some other value of y. Thus if, for a general solution $y = 2x + A$, we have the initial condition that $y = 0$ when $x = 0$ then A is 0 and so the particular solution is $y = 2x$.

Key point

The term solution is used with a differential equation for the relationship between the dependent and independent variables such that the differential equation is satisfied for all values of the independent variable. The general solution consists of a family of equations which satisfy the differential equation; with a first-order differential equation the general solution involves just one arbitrary constant. Given initial or boundary conditions, it is possible to find a value for the arbitrary constant and so obtain a particular solution.

Example

Verify that $y = e^x$ is a particular solution of the differential equation $dy/dx = y$.

If $y = e^x$ then $dy/dx = e^x$. Thus for all values of y we have $dy/dx = y$ and so $y = e^x$ is a solution.

Example

$y = A\,e^x + B\,e^{2x}$ is a general solution of the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Determine the particular solution for the boundary conditions $y = 3$ when $x = 0$ and $dy/dx = 5$ when $x = 0$.

For $y = A\,e^x + B\,e^{2x}$ with $y = 3$ when $x = 0$ we have $3 = A + B$. With $y = A\,e^x + B\,e^{2x}$ we have $dy/dx = A\,e^x + 2B\,e^{2x}$ and thus with $dy/dx = 5$ when $x = 0$ we have $5 = A + 2B$. This pair of simultaneous equations gives $A = 1$ and $B = 2$. Thus the particular solution is:

$$y = e^x + 2\,e^{2x}$$

We can check that this is a valid solution by substituting it in the differential equation:

$$e^x + 8\,e^{2x} - 3(e^x + 4\,e^{2x}) + 2(e^x + 2\,e^{2x}) = 0$$

Problems 5.1

1 Derive differential equations to represent the following situations:

(a) The velocity v of an object of mass m in terms of time t when thrown vertically upwards against air resistance proportional to the square of its velocity.
(b) The displacement x of a mass m on a spring when the mass is pulled down from its equilibrium position and released when there is a damping force proportional to the velocity.
(c) The velocity v of a boat of mass m on still water in terms of time t after the engines are switched off if the drag forces acting on the boat are proportional to the velocity.
(d) The velocity v of an object falling from rest in air if the drag forces are proportional to the square of the velocity.
(e) The intensity I of a beam of light emerging from a block of glass in terms of the thickness x of the glass if the intensity decreases at a rate proportional to the block thickness.

(f) The rate at which the pressure p at the base of a tank changes with time if liquid of density q enters the tank at the volume rate of q_1 and leaves at the rate of q_2.

(g) The height h of a liquid in a tank open to the atmosphere as a function of time t after a leak from the base of the tank occurs.

2 Verify that the following are solutions of the given differential equations:

(a) $y = \cos 2x$ for $\dfrac{d^2y}{dx^2} + 4y = 0$,

(b) $y = 2\sqrt{x} - \sqrt{x}\ln x$ for $4x^2\dfrac{d^2y}{dx^2} + y = 0$,

(c) $y = e^x \cos x$ for $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 2y = 0$,

(d) $y = 2\,e^x + 3x\,e^x$ for $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 0$

3 For the following general solutions of differential equations, verify that they are solutions and determine the particular solution for the given boundary conditions:

(a) $y = A\,e^x + Bx\,e^x$ for:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0,\ y = 0 \text{ and } \frac{dy}{dx} = 1 \text{ at } x = 0,$$

(b) $y = A \sin \omega t + B \cos \omega t$ for:

$$\frac{d^2y}{dt^2} + \omega^2 y = 0,\ y = 2 \text{ and } \frac{dy}{dt} = 1 \text{ at } t = 0,$$

(c) $y = (A + x^2)\,e^{-x}$ for:

$$\frac{dy}{dx} + y = 2x\,e^{-x},\ y = 2 \text{ at } x = 0$$

4 The differential equation relating the deflection y with distance x from the fixed end of a cantilever with a uniformly distributed load is:

$$\frac{d^2y}{dx^2} = -\frac{w}{2EI}(L^2 - 2Lx + x^2)$$

The general solution is given as:

$$y = -\frac{w}{2EI}\left(\tfrac{1}{2}L^2x^2 - \tfrac{1}{3}Lx^3 + \tfrac{1}{12}x^4\right) + Ax + B$$

Verify that this is the general solution and determine the particular solution for $y = 0$ and $dy/dx = 0$ at $x = 0$.

5.2 First-order differential equations

First-order differential equations are often used to model the behaviour of engineering systems. For example, the exponential growth system where the rate of change dN/dt of some quantity is proportional to the quantity N present can be represented by:

$$\frac{dN}{dt} = kN$$

or exponential decay, e.g. radioactivity, where the rate at which a quantity decreases is proportional to the quantity present:

$$\frac{dN}{dt} = -kN$$

Such differential equations are of the form:

$$\frac{dy}{dx} = f(y) \qquad [10]$$

Another form of differential equation is illustrated by the growth of the voltage v_C across a capacitor in an electrical circuit having a capacitor C in series with a resistor R and connected to a step voltage input of V:

$$RC\frac{dv_C}{dt} + v_C = V$$

Such equations are of the form:

$$\frac{dy}{dx} + Py = Q \qquad [11]$$

where P and Q are constants or functions of x.

This section looks at probably the most common method that is used for the solution of such differential equations, the separation of variables, and how it can be used to determine the output of systems which are modelled by first-order differential equations.

Separation of variables

A first-order equation is said to be *separable* if the variables x and y can be separated. To solve such equations we simply separate the variables and then integrate both sides of the equation with respect to x. The following shows solutions of the various forms taken by separable equations:

- **Equations of the form $\frac{dy}{dx} = f(x)$**

  If we integrate both sides of the equation with respect to x:

  $$\int \frac{dy}{dx}\, dx = \int f(x)\, dx$$

This is equivalent to separating the variables and writing:

$$\int dy = \int f(x)\, dx \qquad [12]$$

> **Example**
>
> Solve the differential equation $dy/dx = 2x$.
>
> Separating the variables gives:
>
> $$\int dy = \int 2x\, dx$$
>
> and thus $y = x^2 + A$.

> **Example**
>
> If $dp/dt = (3 - t)^2$, find p in terms of t given the condition that $p = 3$ when $t = 2$.
>
> Separating the variables gives:
>
> $$dp = (3 - t)^2\, dt$$
>
> and so:
>
> $$\int p = \int (3 - t^2)\, dt = \int (9 - 6t + t^2)\, dt$$
>
> $$p = 9t - \frac{6t^2}{2} + \frac{t^3}{3} + C$$
>
> This is the general solution. Using the given conditions that $p = 3$ when $t = 2$ gives:
>
> $$3 = 9(2) - 3(2)^2 + \frac{(2)^3}{3} + C$$
>
> Hence $C = -5.67$ and so the specific solution is:
>
> $$p = 9t - 3t^2 + \frac{t^3}{3} - 5.67$$

- **Equations of the form $\dfrac{dy}{dx} = f(y)$**

  This can be rearranged to give:

  $$\frac{1}{f(y)} \frac{dy}{dx} = 1$$

  Integrating both sides with respect to x:

  $$\int \frac{1}{f(y)} \frac{dy}{dx}\, dx = \int 1\, dx$$

This is equivalent to separating the variables:

$$\int \frac{1}{f(y)}\,dy = \int 1\,dx \qquad [13]$$

> **Example**
>
> Solve the differential equation $dy/dx = 2y$.
>
> Separating the variables gives:
>
> $$\int \frac{1}{y}\,dy = \int 2\,dx$$
>
> Thus $\ln y = 2x + A$. We can write this as $y = e^{2x+A} = e^{2x}e^{A} = B\,e^{2x}$, where B is a constant.

- **Equations of the form $g(y)\dfrac{dy}{dx} = f(x)$**

Integrating both sides of the equation with respect to x gives:

$$\int g(y)\frac{dy}{dx}\,dx = \int f(x)\,dx$$

This is equivalent to:

$$\int g(y)\,dy = \int f(x)\,dx \qquad [14]$$

> **Example**
>
> Solve the differential equation $dy/dx = 2x/y$.
>
> Separating the variables give:
>
> $$\int y\,dy = \int 2x\,dx$$
>
> Thus $\tfrac{1}{2}y^2 = x^2 + A$.

- **Equations of the form $\dfrac{dy}{dx} = f(x)g(y)$**

This can be rearranged and integrated with respect to x to give:

$$\int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

This is equivalent to:

$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx \qquad [15]$$

> **Key point**
>
> To solve first-order differential equations by separation of variables:
>
> 1. Write the differential equation in the form $f(y)\,dy = g(x)\,dx$.
> 2. Solve by integrating both sides of the equation.

Example

Solve the differential equation $dy/dx = 2yx$.

Separating the variables gives:

$$\int \frac{1}{y}\, dy = \int 2x\, dx$$

Thus $\ln y = x^2 + A$.

Equations which are not of any of the above forms may often be put into one of the forms by a *change of variable*. As an illustration, consider the differential equation $dy/dx = y/(y + x)$. This can be written as:

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{y}{x} + 1}$$

If we let $v = y/x$ then $y = vx$ and $dy/dx = v + x\, dv/dx$. Thus the above equation can be written as:

$$v + x\frac{dv}{dx} = \frac{v}{v+1}$$

$$x\frac{dv}{dx} = \frac{v}{v+1} - v = -\frac{v^2}{v+1}$$

$$\frac{v+1}{v^2}\frac{dv}{dx} = -\frac{1}{x}$$

Integrating with respect to x:

$$\int\left(\frac{1}{v} + \frac{1}{v^2}\right)\frac{dv}{dx}\, dx = -\int \frac{1}{x}\, dx$$

This is equivalent to:

$$\int\left(\frac{1}{v} + \frac{1}{v^2}\right) dv = -\int \frac{1}{x}\, dx$$

Hence $\ln v - (1/v) = -\ln x + A$ and so $\ln(y/x) - (x/y) = -\ln x + A$.

Example

Solve the differential equation $dy/dx = \cos^2 y$ if $y = \pi/4$ when $x = 0$.

We can write the equation as: $\sec^2 y \dfrac{dy}{dx} = 1$

Hence, separating the variables gives:

$\int \sec^2 y \, dy = \int 1 \, dx$

and so we have $\tan y = x + A$. Since $y = \pi/4$ when $x = 0$ then $\tan \pi/4 = A$ and so $A = 1$. Thus $\tan y = x + 1$ or $y = \tan^{-1}(x + 1)$.

5.2.1 The responses of first-order systems

This section looks at how, when differential equations are involved in modelling a system, the dynamic responses of systems can be predicted. For example, if the input signal to a measurement system suddenly changes, the output will not instantaneously change to the new value but some time will elapse before it reaches a steady-state value. If the voltage applied to an electrical circuit suddenly changes to a new value, the current in the circuit will not change instantly to the new value but some time will elapse before it reaches the steady new value. If a continually changing signal is applied to a system, the response of the system may lag behind the input. The way in which a system reacts to input changes is termed its *dynamic characteristic*.

First-order systems and step inputs

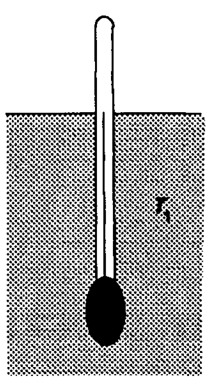


Figure 5.9 *Thermometer inserted in liquid*

Consider a thermometer (Figure 5.9) at temperature T_0 inserted into a liquid at a temperature T_1. We can thus think of the thermometer being subject to a step input, i.e. the input abruptly changes from T_0 to T_1. The thermometer will then, over a period of time, change its temperature until it becomes T_1. Thus we have a measurement system, the thermometer, which has a step input and an output which changes from T_0 to T_1 over some time. How does the output, i.e. the reading of the thermometer T, vary with time.

The rate at which energy enters the thermometer from the liquid is proportional to the difference in temperature between the liquid and the thermometer. Thus, at some instant of time when the temperature of the thermometer is T, we can write:

$$\frac{dQ}{dt} = h(T_1 - T)$$

where h is a constant called the *heat transfer coefficient*. For a thermometer with a specific heat capacity c and a mass m, the relationship between heat input Q and the consequential temperature change is:

$Q = mc$ (temperature change)

When the rate at which heat enters the thermometer is dQ/dt, we can write for the rate at which the temperature changes:

$$\frac{dQ}{dt} = mc\frac{dT}{dt}$$

Thus:

$$mc\frac{dT}{dt} = h(T_1 - T)$$

We can rewrite this with all the output terms on one side of the equals sign and the input on the other, thus:

$$mc\frac{dT}{dt} + hT = hT_1 \qquad [16]$$

We no longer have a simple relationship between the input and output but a relationship which involves time. The form of this equation is typical of first-order systems.

We can solve this equation by separation of the variables:

$$\int \frac{1}{T_1 - T}\,dT = \int \frac{h}{mc}\,dt$$

$$-\ln(T_1 - T) = (h/mc)t + A$$

where A is a constant. This can be rewritten as:

$$T_1 - T = e^{A}\,e^{t/\tau} = C\,e^{t/\tau}$$

where $\tau = mc/h$ and is termed the *time constant*. The time constant can be defined as the value of the time which makes the exponential term become e^{-1}. $T = T_0$ at $t = 0$ and so $C = T_1 - T_0$. Thus:

$$T = T_1 + (T_0 - T_1)\,e^{-t/\tau} \qquad [17]$$

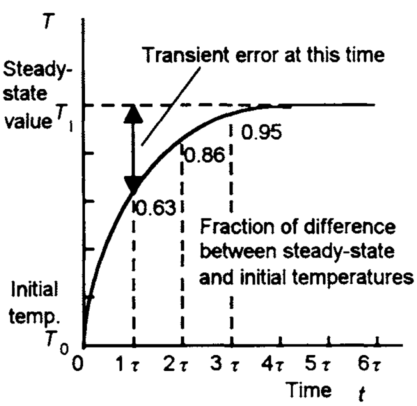


Figure 5.10 *Response of a first-order system to a step input*

The first term is the *steady-state value*, i.e. the value that will occur after sufficient time has elapsed for all transients to die away, and the second term a transient one which changes with time, eventually becoming zero. Figure 5.10 shows graphically how the temperature T indicated by the thermometer changes with time.

After a time equal to one time constant the output has reached about 63% of the way to the steady-state temperature, after a time equal to two time constants the output has reached about 86% of the way, after three time constants about 95% and after about four time constants it is virtually equal to the steady-state value. The error at any instant is the difference between what the thermometer is indicating and what the temperature actually is. Thus:

$$\text{error} = T - T_1 = (T_0 - T_1)\,e^{-t/\tau} \qquad [18]$$

This error changes with time and eventually will become zero. Thus it is a transient error.

If a thermometer is required to be fast reacting and quickly attain the temperature being measured, it needs to have a small time constant. Since $\tau = mc/h$, this means a thermometer with a small mass, a small thermal capacity and a large heat transfer coefficient. If we compare a mercury-in-glass thermometer with a thermocouple, then the smaller mass and specific heat capacity of the thermocouple will give it a smaller time constant and hence a faster response to temperature changes.

Example

A thermometer indicates a temperature of 20°C when it is suddenly immersed in a liquid at a temperature of 60°C. If the thermometer behaves as a first-order system and has a time constant of 5 s what will its readings be after (a) 5 s, (b) 10 s, (c) 15 s.

The temperature T of the thermometer varies with time according to equation [17]:

$$T = T_1 + (T_0 - T_1)\, e^{-t/\tau} = 60 - 40\, e^{-t/5}$$

After 5 s the thermometer reading will have reached about 63% of the way to the steady-state value, after 10 s about 86%, after 15 s about 95% and after 20 s it is virtually at the steady-state value. Thus after 5 s the reading is 45.3°C, after 10 s it is 54.6°C, after 15 s it is 58.0°C.

Example

A thermometer which behaves as a first-order element has a time constant of 15 s. Initially it reads 20°C. What will be the time taken for the temperature to rise to 90% of the steady-state value when it is immersed in a liquid of temperature 100°C, i.e. a temperature of 92°C?

Equation [17], $T = T_1 + (T_0 - T_1)\, e^{-t/\tau}$, can be arranged as:

$$\frac{T - T_1}{T_1 - T_0} = e^{-t/\tau}$$

With $T - T_0$ as 90% of $T_1 - T_0$, then we have $T - T_1$ as 10% of $T_1 - T_0$ and thus:

$$0.10 = e^{-t/15}$$

Taking logarithms gives $-2.30 = -t/15$ and so $t = 34.5$ s.

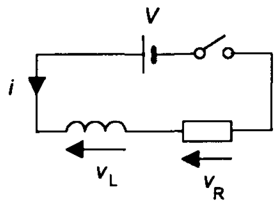


Figure 5.11 *Circuit with series inductance and resistance*

Maths in action

Transients in electrical circuits

Consider the growth of current in a circuit possessing inductance and resistance (Figure 5.11). At some time t after the switch is closed and the current is i, we have:

$$V = v_L + v_R$$

Since $v_R = Ri$ and $v_L = L\,di/dt$ (equation [6]):

$$V = L\frac{di}{dt} + Ri$$

This is a first-order differential equation. It can be solved by separating the variables:

$$\frac{di}{V - Ri} = \frac{dt}{L}$$

If the switch is closed at time $t = 0$ then $i = 0$ when $t = 0$. For the other limit of integration we look for the current to be i at time t. Thus:

$$\int_0^i \frac{1}{V - Ri}\,di = \frac{1}{L}\int_0^t dt$$

$$\left[-\frac{1}{R}\ln(V - Ri)\right]_0^i = \frac{1}{L}[t]_0^t$$

$$-\frac{1}{R}\ln(V - Ri) + \frac{1}{R}\ln V = \frac{t}{L}$$

$$\frac{1}{R}\ln\left(\frac{V}{V - Ri}\right) = \frac{t}{L}$$

$$\ln\left(\frac{V}{V - Ri}\right) = \frac{Rt}{L}$$

$$\frac{V}{V - Ri} = e^{Rt/L} \quad \text{or} \quad \frac{V - Ri}{V} = e^{-Rt/L}$$

$$V - Ri = V\,e^{-Rt/L}$$

$$i = \frac{V}{R}(1 - e^{-Rt/L})$$

The maximum circuit current I is V/R and so:

$$i = I(1 - e^{-Rt/L})$$

When $t = L/R$ then $i = I(1 - e^{-1}) = 0.63I$. This is the same as in Figure 5.10 and so L/R is the time constant of the circuit.

First-order systems in general

In general, a first-order system has a differential equation which can be written in the form:

$$a_1 \frac{dx}{dt} + a_0 x = by \qquad [19]$$

where x is the output, t the time and y the input; a_0, a_1 and b are constants for the system represented by the equation. The left-hand side of the equals sign contains the output related terms and the right-hand side the input related terms. This equation can be rearranged as:

$$\frac{a_1}{a_0}\frac{dx}{dt} + \frac{a_0}{a_0}x = \frac{b}{a_0}y$$

and, if we let $\tau = a_1/a_0$ and $k = b/a_0$, then we have:

$$\tau \frac{dx}{dt} + x = ky \qquad [20]$$

τ defines the *time constant* of the system and k the *static system sensitivity*.

The steady-state value of the output occurs when $dx/dt = 0$ and so $x = ky$ and thus:

$$\text{steady-state output value} = ky \qquad [21]$$

The solution of the differential equation for a step input from some initial value to final value at time $t = 0$ is of the form:

$$x = \text{steady-state value} + (\text{initial value} - \text{steady-state value})\, e^{-t/\tau} \qquad [22]$$

Table 5.1 *First-order system response*

Time	% response	% dynamic error
0	0.0	100.0
1τ	63.2	36.8
2τ	86.5	13.5
3τ	95.0	5.0
4τ	98.2	1.8
5τ	99.3	0.7
∞	100.0	0.0

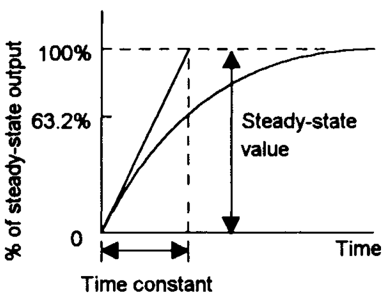


Figure 5.12 *Step response of a first-order system*

Table 5.1 shows the percentage of the response, i.e. (x – initial value)/ (steady – initial values) × 100%, that will have been achieved after various multiples of the time constant. The percentage dynamic error is (steady-state value – x)/(steady – initial value) × 100%. With a step input, the time constant can be defined as the time taken for the output to reach 63.2% of the steady-state value (see Figure 5.10).

There is an alternative way of defining the time constant. At the instant the input starts and we have $t = 0$, then $x = 0$ and so equation [20] gives $\tau\, dx_0/dt = ky$, where dx_0/dt is the initial gradient of the graph of output with time. Thus, since ky is the steady-state value:

$$\text{initial gradient of graph} = (\text{steady-state value})/\tau \qquad [23]$$

Thus, on a graph of output plotted against time for a step input (Figure 5.12), if we draw the tangent to the curve at time $t = 0$, equation [23] gives the initial gradient and so the time constant can be considered to be the time taken for the output to reach the steady-state value if the initial rate of change of output with time were maintained.

Key point

All first-order systems have the input–output relationship defined by a differential equation of the form:

$$\tau \frac{dx}{dt} + x = ky$$

and all give a response to a step input of the form shown in Figure 5.10.

The time constant can be defined as the time taken, when there is a step input, as:

- the output to reach 63.2% of the steady-state value;
- the output to reach the steady-state value if the initial rate of change of output with time were maintained.

Key point

A way of looking at differential equations which you might come across is in terms of the D-operator. The term *operator* is used for a function which transforms one function into another function. The *D-operator* is such a function which is sometimes used with differential equations. With such an operator we regard dy/dx as the result of an operator applied to the function y and write this as Dy.

$$\frac{dy}{dx} = \frac{d}{dx}(y) = D(y)$$

The differential equation:

$$\tau\frac{dx}{dt} + x = ky$$

thus becomes written as:

$$\tau Dx + x = ky$$

D behaves like an ordinary algebraic quantity and so we can write:

$$\frac{\text{output } x}{\text{input } y} = \frac{k}{\tau D + 1}$$

Thus $k/(\tau D + 1)$ is the quantity we operate on the input by in order to give the output and is called a *transfer function*.

Likewise, we have:

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

and can write second-order differential equations in terms of the D-operator and obtain a transfer function.

Example

An electrical circuit consisting of resistance R in series with an initially uncharged capacitor of capacitance C has an input of a step voltage V at time $t = 0$. Determine (a) how the potential difference across the capacitor will change with time and (b) with $R = 1$ MΩ, $C = 4$ μF and a step voltage of 12 V, the potential difference across the capacitor after 2 s.

(a) The differential equation, equation [5], is:

$$RC\frac{dv_C}{dt} + v_C = V$$

This equation is the same form as equation [19] so we can recognise that the solution must be of the form given by equation [21] with the time constant being RC:

x = steady-state value + (initial value − steady-state value) $e^{-t/\tau}$

$$v_C = V + (0 - V)\,e^{-t/\tau} = V(1 - e^{-t/\tau})$$

(b) The time constant is $RC = 1 \times 10^6 \times 4 \times 10^{-6} = 4$ s. Thus after 2 s, $v_C = 12(1 - e^{-2/4}) = 4.72$ V.

Example

A thermocouple in a protective sheath has an output voltage θ_0 in volts related to the input temperature θ_i in °C by the equation:

$$30\frac{d\theta_0}{dt} + 3\theta_0 = 1.5 \times 10^{-5}\theta_i$$

Determine the time constant τ and the static system sensitivity k.

To put the equation in the standard form of equation [20] we divide by 3:

$$10\frac{d\theta_0}{dt} + \theta_0 = 0.5 \times 10^{-5}\theta_i$$

The time constant is thus 10 s and the static system sensitivity 0.5×10^{-5} V/°C. The thermocouple thus takes 10 s to reach 63.2% of its steady-state output and we need to wait at least three times this time for the output to be close to the voltage corresponding to the temperature being measured.

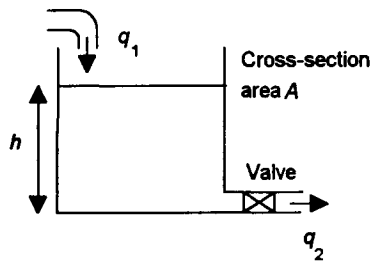


Figure 5.13 *Example*

Example

Determine the time constant and the static system sensitivity for the hydraulic system shown in Figure 5.13 in which a liquid flows into a container at a constant rate and liquid also flows out of the container through a valve at a constant rate.

The differential equation for this system was developed in Chapter 3. The rate of change of liquid volume in the container with time is $A\,\mathrm{d}h/\mathrm{d}t$ and so:

$$q_1 - q_2 = A\frac{\mathrm{d}h}{\mathrm{d}t}$$

For the resistance term for the valve we have $p_1 - p_2 = Rq_1$ and so, since the pressure difference is $h\rho g$:

$$h\rho g = Rq_2$$

Thus, substituting for q_2 gives:

$$q_1 - \frac{h\rho g}{R} = A\frac{\mathrm{d}h}{\mathrm{d}t}$$

and so we can write:

$$A\frac{\mathrm{d}h}{\mathrm{d}t} + \frac{\rho g}{R}h = q_1$$

We can put this equation in the standard form by dividing by $\rho g/R$:

$$\frac{AR}{\rho g}\frac{\mathrm{d}h}{\mathrm{d}t} + h = \frac{R}{\rho g}q_1$$

Comparison with equation [20] thus gives:

time constant $\tau = \dfrac{AR}{\rho g}$

and:

static system sensitivity $k = \dfrac{R}{\rho g}$

Note, that in terms of the *D*-operator, we can write the differential equation as:

$\tau Dh + \mathrm{h} = kq_1$

and so a transfer function of:

$$\frac{h}{q_1} = \frac{k}{\tau D + 1}$$

Problems 5.2

1 Solve, by separation of the variables, the following differential equations:

(a) $\frac{dy}{dx} = \frac{1}{x}$, (b) $\frac{dy}{dx} = \cos \frac{1}{2}x$, (c) $\frac{dy}{dx} = y^2$, (d) $\frac{dy}{dx} = -2y$,

(e) $\frac{dy}{dx} = 2x(y^2 + 1)$, (f) $\frac{dy}{dx} = 3x^2 \, e^{-y}$, (g) $\frac{dy}{dx} = 4 + 3x^2$,

(h) $\frac{dy}{dx} = 2y^2$, (i) $\frac{dy}{dx} = 3x^2 \, e^{-y}$, (j) $\frac{dy}{dx} = \frac{1}{2y}$,

(k) $x^2 \frac{dy}{dx} - y + 1 = 0$, (l) $\frac{dy}{dx} = x^2 y$

2 Determine the solution of the differential equation $dy/dx = 2xy^2$ if $y = ½$ when $x = 0$.

3 A capacitor of capacitance C which has been charged to a voltage V_0 is discharged through a resistance R. Determine how the voltage v_C across the capacitor changes with time t if $dv_C/dt = -V/RC$.

4 The rate at which radioactivity decays with time t is given by the differential equation $dN/dt = -kN$, where N is the number of radioactive atoms present at time t. If at time $t = 0$ the number of radioactive atoms is N_0, solve the differential equation and show how the number of radioactive atoms varies with time.

5 When a steady voltage V is applied to a circuit consisting of a resistance R in series with inductance L, determine how the current i changes with time t if $L\,di/dt + Ri = V$ and $i = 0$ when $t = 0$.

6 Determine the solution of the differential equation $dy/dx = 2 - y$ if $y = 1$ when $x = 0$.

7 A stone freely falls from rest and is subject to air resistance which is proportional to its velocity. Derive the differential equation describing its motion and hence determine how its velocity v varies with time t if $v = 0$ at $t = 0$. Take the acceleration due to gravity as 10 m/s^2.

8 For a belt drive, the difference in tension T between the slack and tight sides of the belt over a pulley is related to the angle of lap θ on the pulley by $dT/d\theta = \mu T$, where μ is the coefficient of friction. Solve the differential equation if $T = T_0$ when $\theta = 0°$.

9 A rectangular tank is initially full of water. The water, however, leaks out through a small hole in the base at a rate proportional to the square root of the depth of the water. If the tank is half empty after one hour, how long must elapse before it is completely empty?

10 For a circuit containing resistance R in series with capacitance C, the potential difference v_C across the capacitor varies with time, being given by $v_C = V - V\,e^{-t/RC}$. What is the time constant for the circuit?

11 A hot object cools at a rate proportional to the difference between its temperature and that of its surroundings. If it initially is at 75°C and cooling at a rate of 2° per minute, what will be its temperature after 15 minutes if the surroundings are at a temperature of 15°C?

12 A sphere of ice melts so that its volume V changes at the rate given by $\mathrm{d}V/\mathrm{d}t = -4\pi kr^2$, where k is a constant and r is the radius at time t after it began to melt. Show that, if R is the initial radius, $r = R\,\mathrm{e}^{-kt}$.

13 A 1000 μF capacitor has been charged to a potential difference of 12 V. At time $t = 0$ it is discharged through a 20 kΩ resistor. What will be the potential difference across the capacitor after 2 s?

14 Determine how the circuit current varies with time when there is a step voltage V input to a circuit having an inductance L in series with resistance R.

15 A sensor behaves as a capacitance of 2 μF in series with a 1 MΩ resistance. As such the relationship between its input y and output x is given by $2(\mathrm{d}x/\mathrm{d}y) + x = y$. How will the output vary with time when the input is a unit step input at time $t = 0$?

16 A system is specified as being first order with a differential equation relating output x to input y by:

$$a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_0x = y$$

If it has a time constant of 10 s and a steady-state value of 5. How will the output of the system vary with time when subject to a step input?

17 A sensor is first order with a differential equation relating its output x for input y by:

$$a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_0x = y$$

If it has a time constant of 1 s, what will be the percentage dynamic error after (a) 1 s, (b) 2 s, from a unit step input signal to the sensor?

18 How long must elapse for the dynamic error of a sensor with a differential equation of the form:

$$a_1\frac{\mathrm{d}x}{\mathrm{d}t} + a_0x = y$$

and subject to a step input to drop below 5% if the sensor is first order with a time constant of 4 s?

19 A thermometer originally indicates a temperature of 20°C and is then suddenly inserted into a liquid at 45°C. The thermometer has a time constant of 2 s. (a) Derive a differential equation showing how the thermometer reading is related to the temperature input and (b) give its solution showing how the thermometer reading varies with time.

5.3 Second-order differential equations

As an example of a second-order ordinary differential equation, consider the displacement y of a freely falling object in a vacuum as a function of time t. It falls with the acceleration due to gravity g and is described by the second-order differential equation:

$$\text{acceleration} = \frac{d^2y}{dt^2} = g \qquad [24]$$

Another example is the displacement y of an object when freely oscillating with simple harmonic motion when there is damping, this being described by the second-order differential equation:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0 \qquad [25]$$

If the oscillating object is not left freely to oscillate when some external force is applied, say $F \sin \omega t$, then we have:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F\sin\omega t \qquad [26]$$

Key point

In general, a second-order differential equation has the form:

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = b$$

where a_2, a_1, a_0 and b are functions of x, b often being termed the forcing function.

With a series electrical circuit containing resistance R, capacitance C and inductance L, the potential difference v_C across the capacitor when it is allowed to discharge is described by the second-order differential equation:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt^2} + v_C = 0 \qquad [27]$$

If such a circuit has a voltage V applied to it we have:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt^2} + v_C = V \qquad [28]$$

Arbitrary constants

Consider an object falling freely with the acceleration due to gravity g. If we take g to be 10 m/s^2 then equation [1] becomes:

$$\frac{d^2y}{dt^2} = 10$$

If we integrate both sides of the equation with respect to t we have:

$$\int \frac{d^2y}{dt^2}\,dt = \int 10\,dt$$

$$\frac{dy}{dt} = 10t + A$$

where A is the constant of integration. If we now integrate this equation with respect to t:

$$\int \frac{dy}{dt}\,dt = \int (10t + A)\,dt$$

$$y = 5t^2 + At + B$$

where B is the constant arising from this integration. Thus the above general solution for the second-order differential equation has two arbitrary constants. With all second-order differential equations there will be two arbitrary constants because two integrations are needed to obtain the solution.

Because there are two arbitrary constants with a second-order differential equation, two sets of values are needed to determine them. This is generally done by specifying two initial conditions: the value of the solution and the value of the derivative at a single point. Thus we might have the initial conditions that $y = 20$ at $t = 0$ and $dy/dt = 0$ at $x = 0$.

Key point

The general solution for a second-order differential equation will have two arbitrary constants.

Example

If the general solution to the differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

is $y = A\,e^x + Bx\,e^x$, determine the solution if $y = 1$ at $x = 0$ and $dy/dx = -1$ at $x = 0$.

From the initial condition $y = 1$ at $x = 0$ we have, when substituting these values in the general solution, $1 = A + 0$. Thus $A = 1$. If we differentiate the general solution to give $dy/dx = A\,e^x + Bx\,e^x + B\,e^x$ and substitute the initial condition $dy/dx = 0$ at $x = 0$, then $-1 = A + 0 + B$ and so $B = -2$. Thus the solution is $y = e^x - 2x\,e^x$.

5.3.1 Second-order homogeneous differential equations

Consider a second-order differential equation of the basic form:

$$a_2\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_0y = 0 \qquad [29]$$

where a_2, a_1 and a_0 are constants. Such a differential equation is said to be *homogeneous* since, when all the dependent variables are moved to the left of the equal sign, there is just a zero on the right. In the case of a homogeneous linear first-order differential equation with constant coefficients:

$$a_1\frac{dy}{dx} + a_0y = 0$$

we have $dy/dx = -(a_0/a_1)y$ and thus, by separation of the variables, the solution is $\ln y = -(a_0/a_1)x + A$ or $y = C\,e^{kx}$, where $k = -(a_0/a_1)$. To solve the constant coefficient second-order differential equation it seems reasonable to consider that it might have a solution of the form $y = A\,e^{sx}$, where A and s are constants. Thus, trying this as a solution, the second-order differential equation [6] becomes:

$$a_2As^2\,e^{sx} + a_1As\,e^{sx} + a_0A\,e^{sx} = 0$$

Since the exponential function is never zero we must have, if $y = A\,e^{sx}$ is to be a solution:

$$a_2s^2 + a_1s + a_0 = 0 \qquad [30]$$

Equation [30] is called the *auxiliary equation* or *characteristic equation* associated with the differential equation [29]. This quadratic equation has the roots:

$$s = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2a_0}}{2a_2} \qquad [31]$$

The roots of the auxiliary equation, as given by equation [31], can be:

- ***Two distinct real roots if $a_1^2 > 4a_2a_0$***

  The general solution to the differential equation is then:

  $$y = A\,e^{s_1x} + B\,e^{s_2x} \qquad [32]$$

  Example

  Determine the general solution of the differential equation:

  $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

  Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

  $$s^2 - s + 6 = 0$$

  which factors as $(s - 3)(s + 2) = 0$ and so $s_1 = 3$ and $s_2 = -2$. Thus the general solution is:

  $$y = A\,e^{3x} + B\,e^{-2x}$$

- ***Two equal real roots if $a_1^2 = 4a_2a_1$***

  This gives $s_1 = s_2 = -a_1/2a_2$. In order to have a solution with two arbitrary constants we *cannot* have a general solution of:

$$y = A\,e^{s_1x} + B\,e^{s_2x} = (A + B)\,e^{sx} = C\,e^{sx}$$

since this can be reorganised to imply only one constant. Thus we try a second solution of the form $y = Bx\,e^{sx}$. Then, since $dy/dx = B\,e^{sx} + Bsx\,e^{sx}$ and $d^2y/dx^2 = 2Bs\,e^{sx} + Bs^2x\,e^{sx}$, substituting into equation [29] gives:

$$a_2(2s + s^2x) + a_1(1 + sx) + a_0x = 0$$

$$(a_2s^2 + a_1s + a_0)x + (2a_2s + a_1) = 0$$

But $a_2s^2 + a_1s + a_0 = 0$ is the auxiliary equation and so the first term is zero. Also $s = -a_1/2a_2$ and so the second term is zero. Thus $y = Bx\,e^{sx}$ is a solution. The general solution is thus:

$$y = A\,e^{s_1x} + Bx\,e^{s_2x} \qquad [33]$$

Example

Determine the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

$$s^2 + 8s + 16 = 0$$

This factors as $(s + 4)(s + 4) = 0$ and so we have two roots of $s = -4$. The solution is thus of the form given in equation [33]:

$$y = A\,e^{-4x} + Bx\,e^{-4x}$$

- ***Two distinct complex roots if $a_1^2 < 4a_2a_0$***
  With this condition, equation [8] can be written as:

$$s = \frac{-a_1 \pm j\sqrt{4a_2a_0 - a_1^2}}{2a_2} = \alpha \pm j\beta$$

where $\alpha = -(a_1/2a_2)$ and $\beta = \sqrt{(a_0/a_2) - (a_1/a_2)^2/4}$. Thus the general solution is:

$$y = A\,e^{(\alpha + j\beta)x} + B\,e^{(\alpha - j\beta)x}$$

This can be written as:

$$y = A\,e^{\alpha x}\,e^{j\beta x} + B\,e^{\alpha x}\,e^{-j\beta x} = e^{\alpha x}(A\,e^{j\beta x} + B\,e^{-j\beta x})$$

There is a relationship called Euler's formula which enables the above equation to be written as:

$$y = e^{\alpha x}[A(\cos\beta x + j\sin\beta x) + B(\cos\beta x - j\sin\beta x)]$$

$$= e^{\alpha x}[(A+B)\cos\beta x + j(A-B)\sin\beta x]$$

$$= e^{\alpha x}[C\cos\beta x + D\sin\beta x] \qquad [34]$$

Key point

Euler's equation

A complex number z can be expressed as:

$z = |z|(\cos\theta + j\sin\theta)$

Hence:

$$\frac{dz}{d\theta} = |z|(-\sin\theta + j\cos\theta) = j(\cos\theta + j\sin\theta)$$

But this means that the derivative is just j times z. A function with this property that the derivative is proportional to itself is the exponential. Thus we can write:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This is known as Euler's equation.

An alternative way of arriving at this equation is to consider sines and cosines expressed as series, and so write:

$$z = |z|\left(1 - \frac{\theta^2}{2!} + \frac{\theta^2}{4!} - \dots\right) + |z|j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

Since $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = j$, etc. we can write the equation as:

$$z = |z|\left(1 + j\theta + \frac{j^2\theta^2}{2!} + \frac{j^3\theta^3}{3!} + \frac{j^4\theta^4}{4!} + \dots\right)$$

But this is the form of the series for e^x. Thus we can write:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

Example

Determine the general solution of the differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

and the particular solution if $y = 1$ and $dy/dx = 2$ at $x = 0$.

Trying $y = A\,e^{sx}$ as a solution gives, since $dy/dx = sA\,e^{sx}$ and $d^2y/dx^2 = s^2A\,e^{sx}$, the auxiliary equation:

$$s^2 - 2s + 5 = 0$$

This has roots:

$$s = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm j2$$

The general solution will thus be of the form given for equation [34]:

$$y = e^x(C\cos 2x + D\sin 2x)$$

With $y = 1$ when $x = 0$ we have:

$$1 = 1(C + 0)$$

and thus $C = 1$. Differentiating the solution gives:

$$\frac{dy}{dx} = e^x(2C\sin 2x - 2D\cos 2x) + e^x(C\cos 2x + D\sin 2x)$$

With $dy/dx = 2$ at $x = 0$ we have:

$$2 = -2D + C$$

and so $D = -½$. Thus the particular solution is:

$$y = e^x(\cos 2x + ½\sin 2x)$$

Key points

If the roots of the auxiliary equation are both real, i.e. $a_1^2 > 4a_2a_0$, then:

$$y = A\,e^{s_1x} + B\,e^{s_2x}$$

If the roots are real and equal, i.e. $a_1^2 = 4a_2a_0$

$$y = A\,e^{s_1x} + Bx\,e^{s_2x}$$

If the roots are imaginary, i.e. $a_1^2 < 4a_2a_0$, then:

$$y = e^{\alpha x}[C\cos\beta x + D\sin\beta x]$$

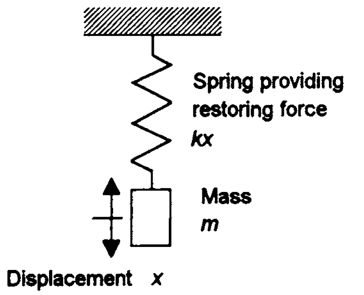


Figure 5.14 *Mass on a spring*

Maths in action

System of a mass on a spring

Consider engineering systems which can be represented by a mass on a spring (Figure 5.14); we will assume there is no damping. If the mass is pulled downwards and then released, it oscillates on the spring. The force acting on the mass is just the restoring force and so:

$$F = m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx = 0$$

This is a homogeneous second-order differential equation. If we try the solution $x = A\,e^{t}$, then we obtain:

$$mAs^2 + kA = 0$$

and so $s^2 = -k/m$ and we can write:

$$s = \pm j\sqrt{\frac{k}{m}}$$

with s an imaginary quantity. If we let $\omega = \sqrt{(k/m)}$, then the solution to the differential equation:

$$x = A\,e^{+j\omega t} + B\,e^{-j\omega t}$$

which we can write as:

$$x = C\cos \omega t + D\sin \omega t$$

If $x = 0$ when $t = 0$ then $C = 0$ and thus:

$$x = D\sin \omega t$$

The oscillations can be described by a sine function with angular frequency ω and amplitude D.

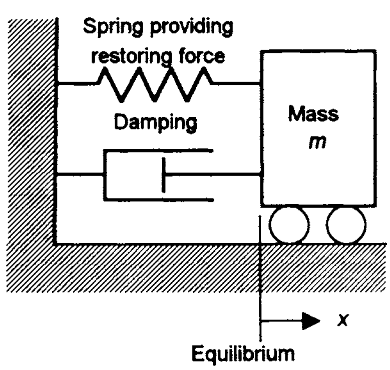


Figure 5.15 *Mass, spring, damper system. Note that the mass is considered to be on rollers so that we can neglect friction*

Maths in action

System of a damped mass and a spring

In a similar manner to the previous Maths in action, we may consider the oscillations of the damped system shown in Figure 5.15. The mass is constrained to move in purely a horizontal motion so we need only consider horizontal forces. There are many engineering systems which can be modelled by such a system. Later in this section we look at this system when there is an external force acting on the mass.

The force resulting from compressing the spring is proportional to the change in length x of the spring, i.e. kx with k being a constant termed the spring stiffness. The force arising from the damping is proportional to the rate at which the displacement of the piston is changing, i.e. $c\,dx/dt$ with c being a constant. Thus:

$$\text{net force applied to mass} = -kx - c\frac{dx}{dt}$$

This net force will cause the mass to accelerate. Thus:

$$m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}$$

We can write this as:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

In the absence of damping we have $m\,d^2x/dt^2 + kx = 0$ and the spring *naturally* oscillating (see Earlier Maths in action in this chapter) with an angular frequency, which we can call the natural angular frequency of ω_n given by:

$$\omega_n = \sqrt{\frac{k}{m}}$$

If we define a constant ζ, termed the *damping ratio*, by:

$$\zeta = \frac{c}{2\sqrt{mk}}$$

then we can write the second-order differential equation as:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2x = 0$$

Now, in order to solve this differential equation, we use a technique similar to that detailed in the previous Maths in action. We will try a solution of the form $x = A\,e^{st}$. This produces the auxiliary equation:

$$As^2e^{st} + 2\zeta\omega_n sA\,e^{st} + \omega_n^2A\,e^{st} = 0$$

and so:

$$s^2 + 2\omega_n\zeta s + \omega_n^2 = 0$$

This is a quadratic and we can use the usual equation for the roots of a quadratic to obtain:

$$s = \frac{-2\omega_n\zeta \pm \sqrt{4\omega_n^2\zeta^2 - 4\omega_n^2}}{2} = -\omega_n\zeta \pm \omega_n\sqrt{\zeta^2 - 1}$$

The general solution is thus:

$$x = A_1\,e^{s_1 t} + A_2\,e^{s_2 t}$$

The resulting oscillation of the system depends on the term inside the square root sign.

Damping ratio with a value between 0 and 1
This gives two complex roots:

$$s = -\omega_n\zeta \pm j\sqrt{1-\zeta^2}$$

If we let $\omega = \omega_n\sqrt{1-\zeta^2}$ then $s = -\omega_n\zeta \pm j\omega$ we obtain:

$$x = (A_1 e^{j\omega t} + A_2 e^{-j\omega t})\,e^{-\zeta\omega_n t}$$

By using Euler's equation (see Key point earlier in this section) we can write this as:

$$x = e^{-\zeta\omega_n t}(P\cos\omega t + Q\sin\omega t)$$

The exponential term means we have a damped oscillation. The equation can be expressed in an alternative form, since for the sine of a sum we can write $\sin(\omega t + \phi) = \sin\omega t\cos\phi + \cos\omega t\sin\phi$. If we let P and Q represent the opposite sides of a right-angled triangle of angle ϕ (Figure 5.16), then $\sin\phi = P/\sqrt{(P^2 + Q^2)}$ and $\cos\phi = Q/\sqrt{(P^2 + Q^2)}$ and so:

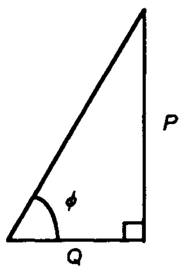


Figure 5.16 *Angle ϕ*

$$x = \sqrt{P^2 + Q^2}\;e^{-\zeta\omega_n t}\sin(\omega t + \phi)$$

$$x = C\,e^{-\zeta\omega_n t}\sin(\omega t + \phi)$$

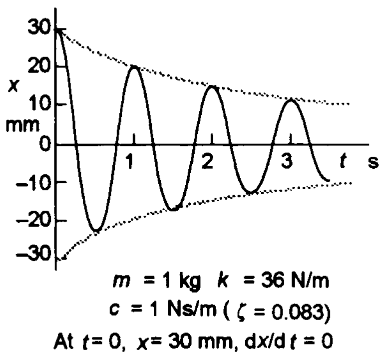


Figure 5.17 *Under-damped oscillation*

where C is a constant and ϕ a phase difference. This describes a sinusoidal oscillation which is damped, the exponential term being the damping factor which gradually reduces the amplitude of the oscillation (Figure 5.17). Such a motion is said to be *under-damped.*

Damping ratio with the value 1
This gives two equal roots $s_1 = s_2 = -\omega_n$ and thus:

$$x = (At + B)\,e^{-\omega_n t}$$

where A and B are constants. This describes a situation where no oscillations occur but x exponentially changes with time. Such a motion is said to be *critically damped.*

Damping ratio greater than 1
This gives two real roots $s_1 = -\omega_n\zeta + \omega_n\sqrt{(\zeta^2 - 1)}$ and $s_2 = -\omega_n\zeta - \omega_n\sqrt{(\zeta^2 - 1)}$, thus:

$$x = A_1\,e^{s_1 t} + A_2\,e^{s_2 t}$$

This describes a situation where no oscillations occur but x exponentially changes with time, taking longer to reach the steady-state zero displacement value than the critically damped motion. Such a motion is said to be *over-damped*.

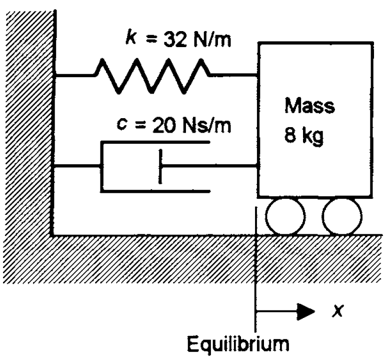


Figure 5.18 *Example*

Example

This example illustrates the discussion in the above Maths in action. For the system shown in Figure 5.18, the 8 kg mass is moved 0.2 m to the right of the equilibrium position and released from rest at time $t = 0$. Determine its displacement at time $t = 2$ s.

First we consider whether the system is underdamped, critically damped or overdamped.

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{20}{2\sqrt{8 \times 32}} = 0.625$$

Since the damping factor is less than 1 the system is underdamped. The natural frequency is:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{8}} = 2 \text{ rad/s}$$

and so the undamped frequency is:

$$\omega = \omega_n\sqrt{1 - \zeta^2} = 2\sqrt{1 - (0.625)^2} = 1.561 \text{ rad/s}$$

The motion of the underdamped mass is described by:

$$x = C\,e^{-\zeta\omega_n t}\sin(\omega t + \phi) = C\,e^{-1.25t}\sin(1.561t + \phi)$$

At $t = 0$, we have $x = 0.2$ and so $0.2 = C \sin \phi$. Its velocity v is dx/dt:

$$\frac{dx}{dt} = -1.25C\,e^{-1.24t}\sin(1.561t + \phi)$$
$$+ 1.561C\,e^{-1.25t}\cos(1.561t + \phi)$$

At $t = 0$ the velocity is 0 and so $0 = -1.25C \sin \phi + 1.561C \cos \phi$. We can solve these two simultaneous equations to give $C = 0.256$ m and $\phi = 0.896$ rad.

The displacement x at time t is thus given by:

$x = 0.256\ e^{-1.25t} \sin(1.561t + 0.896)$

Thus, at time t = 2 s we have:

$x = 0.256\ e^{-2.5} \sin(3.122 + 0.896) = -0.0162$ m

The minus sign indicates that the displacement is to the left of the equilibrium position.

5.3.2 Second-order non-homogeneous differential equations

Consider a non-homogeneous linear second-order differential equation with constant coefficients a_2, a_1 and a_0 with $f(x)$ being some function of x, often being referred to as the *forcing function*, applied to the system:

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = f(x) \quad [35]$$

Key point

To determine the solution of an nonhomogeneous second-order differential equation:

1. Find the general solution of the corresponding homogeneous differential equation. This is called the complimentary function.
2. Then add to it any solution which fits the nonhomogeneous differential equation. This is called the particular integral.

With such a non-homogeneous differential equation there is a general solution which is equal to the sum of, what are called, the *complementary function* y_c and the *particular integral* y_p.

$y = y_c + y_p$

The complementary function is obtained by solving the equivalent homogeneous differential equation, i.e. with $f(x) = 0$, and the particular integral by considering the form of the $f(x)$ function and trying a particular solution of a similar form but which contains undetermined coefficients.

Right-hand side of non-homogeneous equation	Trial function, with A, B, C, etc. being undetermined coefficients
Constant	A
Polynomial	$A + Bx + Cx^2 + \ldots$
Exponential	$A\ e^{kx}$
Sine or cosine	$A \sin kx + B \cos kx$

Note: if the right-hand side is a sum of more than one term then the trial solution is the sum of the trial functions for these terms.

Example

Determine the general solution of the differential equation:

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2$$

To obtain the complementary function we consider the equivalent homogeneous differential equation, i.e.

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

$$s^2 - 5s + 6 = 0$$

This can be factored as $(s - 3)(s - 2) = 0$ and so $s_1 = 3$ and $s_2 = 2$. The complementary function is thus:

$$y_c = A\,e^{3x} + B\,e^{2x}$$

To find the particular integral with x^2 we try a solution of the form $y = C + Dx + Ex^2$. This gives $dy/dx = D + 2Ex$ and $d^2y/dx^2 = 2E$. Substituting into the non-homogeneous differential equation gives:

$$2E - 5(2Ex + D) + 6(C + Dx + Ex^2) = x^2$$

Equating coefficients of x^2 gives $6E = 1$ and so $E = 1/6$. Equating coefficients of x gives $-10E + 6D = 0$ and so $D = 10/36 = 5/18$. Equating constants gives $2E - 5D + 6C = 0$ and so $C = 19/108$. Thus the particular integral is:

$$y_p = \tfrac{19}{108} + \tfrac{5}{18}x + \tfrac{1}{6}x^2$$

and so the general solution is:

$$y = y_c + y_p = A\,e^{3x} + B\,e^{2x} + \tfrac{19}{108} + \tfrac{5}{18}x + \tfrac{1}{6}x^2$$

Example

Determine the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2x = 3\,e^{2x}$$

The corresponding homogeneous differential equation is:

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

$s^2 + s - 2 = 0$

This can be factored as $(s + 2)(s - 1) = 0$ and so the roots are $s_1 = -2$ and $s_2 = 1$ and the complementary function is:

$y_c = A\,e^{-2x} + B\,e^x$

For the particular integral with an exponential forcing function we try a solution of the form $y = C\,e^{kx}$. Substituting this into the non-homogeneous differential equation gives:

$k^2C\,e^{kx} + kC\,e^{kx} - 2C\,e^{kx} = 3\,e^{2x}$

Thus we must have $k = 2$ for equality of the exponentials and for the coefficients $(k^2 + k - 2)C = 3$ and hence $C = ¾$. Hence the particular integral is $y_p = ¾\,e^{2x}$ and the general solution is:

$y = y_c + y_p = A\,e^{-2x} + B\,e^x + ¾\,e^{2x}$

Example

Determine the general solution of the differential equation:

$$3\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=2\cos x$$

The corresponding homogeneous differential equation is:

$$3\frac{d^2y}{dx^2}+\frac{dy}{dx}-2y=0$$

Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

$3s^2 + s - 2 = 0$

This can be factored as $(3s - 2)(s + 1) = 0$ and so the roots are $s_1 = 2/3$ and $s_2 = -1$ and the complementary function is:

$y_c = A\,e^{2x/3} + B\,e^{-x}$

For the particular integral we try a solution of the form $y = C\cos kx + D\sin kx$. Substituting this into the non-homogeneous differential equation gives:

$$3(-C\cos kx - D\sin kx) + (-C\sin kx + D\cos kx) - 2(C\cos kx + D\sin kx) = 2\cos x$$

For equality of the cosines we must have $k = 1$ and $-3C + D - 2C = 2$. Equating coefficients of the sines gives $-3D - C - 2D = 0$. Thus we have $C = -5/13$ and $D = 1/13$. The particular integral is thus:

$$y_p = -\tfrac{5}{13}\cos x + \tfrac{1}{13}\sin x$$

The general solution is thus:

$$y = y_c + y_p = A\,e^{2x/3} + B\,e^{-x} - \tfrac{5}{13}\cos x + \tfrac{1}{13}\sin x$$

Exceptional cases of particular integrals

There are situations when the obvious form of function to be tried to obtain the particular integral yields no result because when it is substituted in the differential equation we obtain $0 = 0$. This occurs when the right-hand side of the non-homogeneous differential equation consists of a function that is also a term in the complementary function. To illustrate this, consider the differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-2x}$$

The complementary function is $y = A\,e^{-2x} + B\,e^{x}$. For the particular integral, if we try the solution $y = A\,e^{kx}$ we obtain:

$$4A\,e^{kx} - 2A\,e^{kx} - 2A\,e^{kx} = e^{-2x}$$

and so no solution for A. In such cases we have to try something different.

The basic rule is to multiply the trial solution by x.

Thus we try $y = Ax\,e^{kx}$. This gives, for the above differential equation:

$$(-2A\,e^{kx} + 4Ax\,e^{kx} - 2A\,e^{kx}) + (A\,e^{kx} - 2Ax\,e^{kx}) - 2Ax\,e^{kx} = e^{-2x}$$

Thus $k = -2$ and $-2A + 4Ax - 2A + A - 2Ax - 2Ax = 1$. Equating constants gives $3A = 1$, equating the x coefficients gives $0 = 0$, and so the particular integral is:

$$y_p = \tfrac{1}{3}\,e^{-2x}$$

Example

Determine the general solution of the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 4 - e^{-2x}$$

The corresponding homogeneous differential equation is:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 10y = 0$$

Trying $y = A\,e^{sx}$ as a solution gives the auxiliary equation:

$s^2 - 3s - 10 = 0$

This can be factored as $(s - 5)(s + 2) = 0$ and so the complementary function is $y_c = A\,e^{5x} + B\,e^{-2x}$. The right-hand side of the non-homogeneous differential equation is the sum of two terms for which the trial functions would be C and $Dx\,e^{kx}$. We thus try the sum of these. Thus:

$$Dk^2x\,e^{kx} + Dk\,e^{kx} + Dk\,e^{kx} - 3Dkx\,e^{kx} - 3D\,e^{kx} \\ - 10(C + Dx\,e^{kx}) = 4 - e^{-2x}$$

Equating exponential terms gives $k = -2$, $4Dx - 2D - 2D + 6Dx - 3D - 10Dx = -1$ and so $D = 1/7$. Equating constants gives $-10C = 4$ and so $C = -4/10$. Thus the particular integral is $-(4/10) + (1/7)\,e^{2x}$. The general solution is therefore:

$$y = A\,e^{5x} + B\,e^{-2x} - \tfrac{4}{10} + \tfrac{1}{7}x\,e^{-2x}$$

Forced oscillations of elastic systems

Oscillations of elastic systems in which the system is free to adopt its own frequency of oscillation are said to be natural or free oscillations. When a system is forced to oscillate by some external force F at the frequency of this force, then the oscillations are said to be forced. In general, a simple model of such oscillations is given by a second-order differential equation of the form:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2x = kF \qquad [36]$$

where F is the externally applied force, k a constant, x the system output, ω_n the natural angular frequency and ζ the damping ratio. Steady-state conditions occur when dx/dt and d^2x/dt^2 are zero and so we then have $\omega_n{}^2x_s = kF$.

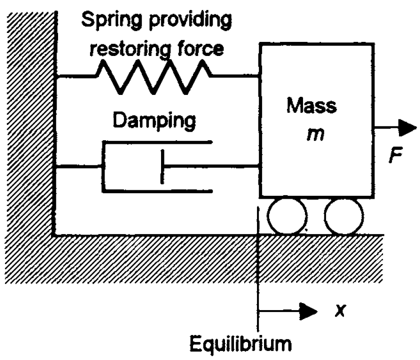


Figure 5.19 *Mass, spring, damper system*

Maths in action

System of a damped mass on a spring

There are many engineering systems which can be modelled by the lumped system of damped mass on a spring and then subject to some externally applied force. An example of a measurement system which can be modelled in this way is a diaphragm pressure gauge. Figure 5.19 illustrates the basic features of such systems.

The net force applied to the mass is the applied force F minus the force resulting from the compressing, or stretching, of the spring and the force from the damper:

$$\text{net force applied to mass} = F - kx - c\frac{dx}{dt}$$

This net force will cause the mass to accelerate. Thus:

$$m\frac{d^2x}{dt^2} = F - kx - c\frac{dx}{dt}$$

We can write this as:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F$$

In the absence of damping and a force F, we have $m\,d^2x/dt^2 + kx = 0$ and the spring *naturally* oscillating (see Earlier Maths in action in this chapter) with an angular frequency, which we can call the natural angular frequency of ω_n given by:

$$\omega_n = \sqrt{\frac{k}{m}}$$

If we define a constant ζ, termed the *damping ratio*, by:

$$\zeta = \frac{c}{2\sqrt{mk}}$$

then we can write the differential equation as:

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2x = \frac{F}{m}$$

Consider a step input such that the applied force jumps from zero to F at time $t = 0$. We can solve the differential equation by determining the complementary function and the particular integral. For the homogeneous form of the differential equation we try a solution of the form $x = A\,e^{st}$ (see earlier Maths in action in this chapter). We thus have the homogeneous equation solutions:

Damping ratio less than 1, i.e. underdamped

$$x = C\,e^{-\zeta\omega_n t}\sin(\omega t + \phi)$$

Damping ratio with the value 1, i.e. critically damped

$$x = (At + B)\,e^{-\omega_n t}$$

Damping ratio greater than 1, i.e. critically damped

$$x = A\,e^{s_1 t} + B\,e^{s_2 t}$$

When we have a step input then we can try for the particular integral $x = A$. Substituting this into the differential equation gives $0 + 0 + A = F/m$. Thus the particular integral is $x = F/m$ and the solutions for the different degrees of damping are:

Under-damped: $x = C\,e^{-\zeta\omega_n t}\sin(\omega t + \phi) + \frac{F}{m}$

Critically damped: $x = (At + B)\,e^{-\omega_n t} + \frac{F}{m}$

Over-damped: $x = A\,e^{s_1 t} + B\,e^{s_2 t} + \frac{F}{m}$

As t tends to an infinite value, in all cases the response tends to a steady-state value of F/m. Figure 5.20 shows the form the solution of the second-order differential equation takes for different values of the damping ratio.

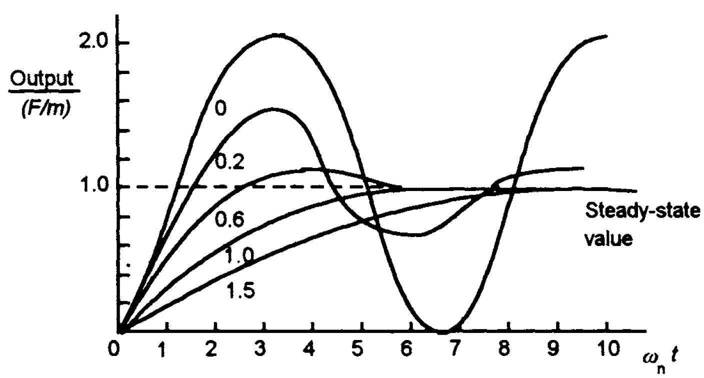


Figure 5.20 *Response of second-order system to step input for different damping factors. The output is plotted as a multiple of the steady-state value F/m. Instead of just giving the output variation with time t, the axis used is $\omega_n t$. This is because t and ω_n always appear as the product $\omega_n t$ and using this product makes the graph applicable for any value of ω_n.*

Example

The dynamic performance of a piezoelectric accelerometer is described by the following second-order differential equation:

$$\frac{d^2\theta_0}{dt^2} + 3\times10^3\frac{d\theta_0}{dt} + 22.5\times10^9\theta_0 = 110\times10^9\theta_i$$

where θ_0 is the output charge in pC and θ_i is the input acceleration in m/s^2. Determine the natural angular frequency, the damping factor and the static system sensitivity.

We can compare the differential equation with the standard form of equation [36] for the oscillations of an elastic system. We thus have:

$22.5 \times 10^9 = \omega_n^2$

and so $\omega_n = 150 \times 10^3$ rad/s. Since $\omega_n = 2\pi f_n$ then the natural frequency $f_n = 150 \times 10^3/2\pi = 23.87$ kHz. We also have:

$3 \times 10^3 = 2\zeta\omega_n$

and so $\zeta = 3 \times 10^3/(2 \times 150 \times 10^3) = 0.01$. The oscillation is thus underdamped.

Steady state occurs when $\omega_n^2 x_s = kF$ and so the static system sensitivity is $x_s/F = k/\omega_n^2 = 110 \times 10^9/(22.5 \times 10^9) = 4.89$ pC/(m/s^2).

Problems 5.3

1 Determine the unique solutions for the following differential equations given the general solutions and initial conditions:

(a) $\frac{d^2y}{dx^2} = 3$, $y = \frac{3}{2}x^2 + Ax + B$, $y = 2$ and $\frac{dy}{dx} = 4$ at $x = 0$,

(b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$, $y = A\,e^{2x} + B\,e^{-3x}$
$y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$,

(c) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$, $y = A\,e^{-2x} + Bx\,e^{-2}$
$y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$,

(d) $\frac{d^2y}{dx^2} - y = 0$, $y = A\,e^{x} + B\,e^{-x}$,
$y = 0$ and $\frac{dy}{dx} = 5$ at $x = 0$,

(e) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, $y = A\,e^{x} + B\,e^{2x}$
$y = 1$ and $\frac{dy}{dx} = 0$ at $x = 0$,

(f) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$, $y = A\,e^{-x} + Bx\,e^{-x}$
$y = 2$ and $\frac{dy}{dx} = -1$ at $x = 0$

2 Determine the general solutions of:

(a) $2\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$, (b) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$,

(c) $\frac{d^2y}{dx^2} - 10\frac{dy}{dx} + 25y = 0$, (d) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$,

(e) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$, (f) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 10y = 0$,

(g) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 5y = 0$, (h) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 8y = 0$

3 Determine the particular solutions of:

(a) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 0$, $y = 0$ and $\frac{dy}{dx} = 3$ at $x = 0$,

(b) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$, $y = 3$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(c) $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0$, $y = 2$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(d) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$, $y = 3$ and $\frac{dy}{dx} = 2$ at $x = 0$,

(e) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$, $y = 1$ and $\frac{dy}{dx} = 1$ at $x = 0$,

(f) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$, $y = 1$ and $\frac{dy}{dx} = 7$ at $x = 0$

4 Determine the general solutions of:

(a) $\frac{d^2y}{dx^2} - 4y = 2\,e^{3x}$, (b) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 3x + 2$,

(c) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 3\cos x$, (d) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{x}$,

(e) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 2e^{x} - 3e^{-x}$,

(f) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 5\sin 2x$,

(g) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3 - 2x^2$

5 Determine the particular solution of the following differential equation if $y = -2$ and $dy/dx = -3$ when $x = 0$:

$$\frac{d^2y}{dx^2} - 4y = 5\,e^{3x}$$

6 Determine the particular solution of the following differential equation if $y = 1$ and $dy/dx = 0$ when $x = 0$:

$$\frac{d^2y}{dx^2} + 9y = \sin 2x$$

7 An object of mass 1 kg is suspended from a rigid support by a vertical spring of stiffness 4 N/m. Determine how the displacement of the object varies with time when the object is pulled down from its initial position and released to freely move if the object is subject to a damping force of five times its velocity?

8 An object of mass 1 kg is suspended from a rigid support by a vertical spring of stiffness 9 N/m. The object is pulled down for an initial displacement of 0.2 m and then released with zero initial velocity. Determine how the displacement of the object varies with time when (a) there is no damping, (b) the damping is twice the velocity of the object.

9 A second-order system has a natural angular frequency of 2.0 rad/s and a damped angular frequency of 1.8 rad/s. What is the damping factor?

10 Determine the natural angular frequency and damping factor for a second-order system with input y and output x described by the following differential equation:

$$0.02\frac{d^2x}{dt^2} + 0.20\frac{dx}{dt} + 0.50x = y$$

11 A sensor can be considered to be a mass-damper-spring system with a mass of 10 g and a spring of stiffness 1.0 N/mm. Determine the natural angular frequency and the damping constant required for the damping element if the system is to be critically damped.

12 Determine whether the system described by the following differential equation is under-damped, critically damped or over-damped when subject to a step input y:

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = y$$

13 An object of mass 1 kg is suspended from a rigid support by a vertical spring of stiffness 9 N/m. What is the damping force per unit velocity which would be needed to give critical damping?

14 Determine the natural angular frequency and damping force per unit velocity for a system having its displacement x with time t described by the following second-order differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 7x = 0$$

15 An object of mass 1 kg is suspended from a rigid support by a vertical spring of stiffness 9 N/m. If there is a damping force of $1v$ opposing the motion of the object, where v is the velocity, determine how the displacement varies with time when the object is given an initial displacement of 0.2 m and an initial velocity of –0.3 m/s.

16 The angular displacement θ of a door controlled by a hydraulic damping mechanism is described by the differential equation:

$$\frac{d^2\theta}{dt^2} + 5\frac{d\theta}{dt} + 4\theta = 0$$

Determine how the angular displacement varies with time t when there is an initial displacement of $\pi/3$ and zero initial angular velocity.

17 An electrical circuit having resistance R, inductance L and capacitance C in series with a step voltage source V at t = 0 has the potential difference across the capacitor v_C described by the differential equation:

$$LC\frac{d^2v_C}{dt^2} + RC\frac{dv_C}{dt} + v_C = V$$

Show that the three possible solutions are:

$$v = A\,e^{s_1t} + B\,e^{s_2t} + V,\ s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$v = (A + Bt)\,e^{-Rt/2L} + V$$
$$v = e^{-Rt/2L}(A\cos\omega t + B\sin\omega t) + V,\ \omega = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$